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ON THE REAL ROOTS OF A RANDOM ALGEBRAIC POLYNOMIAL

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Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of identically distributed independent random variables belonging to the domain of attraction of the symmetric stable law (Gnedenko and Kolmogorov, p. 171). Suppose $a_0, a_1, \cdots a_n$ are non-zero real numbers and $\max_{0 \leqslant r \leqslant n} |a_r| = k_n$, $\min_{0 < r \leqslant n} |a_r| = t_n$ and $\frac{k_n}{t_n} = O$ $(\log n)$. If $N_n(\omega)$ be the number of real roots of the equation $\sum_{r=0}^{n} a_r X_r x^r = 0$, then for all $n > n_0$, it is proved that

$$P\left\{\omega: N_n(\omega) < \frac{\log n}{32 \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)}\right\}$$

$$< \frac{161\ e \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)}{\log n}.$$

INTRODUCTION

§1. The object of this paper is to find the lower bound of $N_n(\omega)$ of the random polynomial

$$f(x) = \sum_{r=0}^{n} a_r X_r(\omega) x^r \qquad \ldots(1)$$

when the coefficients are not identically distributed and belong to the domain of attraction of the symmetric stable law with index 'α' i. e. $0 < \alpha \leqslant 2$. The problem of finding bounds for $N_n(\omega)$ has been considered by various authors. Such type of equation has been considered by Dunnage[3]. Samal and Mishra[5] have considered the same case when the variance is infinite. Mishra, *et al.*[6] considered the lower bound for $N_n(\omega)$ when the coefficients X_r's belong to the domain of attraction of the normal law. In this paper we will extend the results of Mishra *et al.*[6] to the case of domain of attraction of the symmetric stable law.

Theorem—Let $f(x) = \sum_{r=0}^{n} a_r X_r x^r$ be a polynomial of degree n where X_r's are identically distributed independent random variables belonging to the domain of attraction of the symmetric stable law with characteristic function $\exp\{-|t|^{\alpha} h(t)\}$,

$0 < \alpha \leqslant 2$, $h(t)$ being a positive slowly varying function in the neighbourhood of the origin. Let $a_0, a_1, a_2, \ldots a_n$ be non-zero real numbers. Then there exists a positive integer n_0 such that for all $n > n_0$,

$$P\left\{\omega : N_n(w) < \frac{\log n}{32 \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)}\right\}$$

$$< \frac{161\, e \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)}{\log n}$$

provided $\lim\limits_{n\to\infty}\left(\frac{k_n}{t_n}\right)$ is finite, where $k_n = \max\limits_{0<r\leqslant n} |a_r|$ and $t_n = \min\limits_{0<r\leqslant n} |a_r|$.

§2. For the proof of the theorem we need the following definitions, notations and lemmas in the sequel.

Let M be the integer defined by

$$M = \left[\left(\frac{k_n}{t_n}\right)^{\alpha}(\log n)^5\right] + 1 \qquad \ldots(2)$$

where $[x]$ denotes the greatest integer $\leqslant x$; and let k be determined by

$$M^{2k} \leqslant n \leqslant M^{2k+2}. \qquad \ldots(3)$$

From (2) and (3) it follows that

$$\frac{\log n}{4 \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)} \leqslant k \leqslant \frac{\log n}{2 \log\left(\left(\frac{k_n}{t_n}\right)(\log n)^{5/\alpha}\right)}. \qquad \ldots(4)$$

We consider $f(x) = \sum\limits_{r=0}^{n} a_r X_r x^r$ at the points

$$x_m = \left(1 - \frac{1}{M^{2m}}\right)^{1/\alpha} \qquad \ldots(5)$$

for $m = \left[\frac{k}{2}\right] + 1, \left[\frac{k}{2}\right] + 2, \ldots k.$

There are $k/2$ points if k is even and $(k+1)/2$ points if k is odd. We express $f(x_m)$ for sufficiently large n as sum of the three parts as follows :

$$f(x_m) = \left(\sum_1 + \sum_2 + \sum_3\right) a_r X_r(\omega)\, x_m^r$$

where the index r ranges from $M^{2m-1} + 1$ to M^{2m+1} in $\sum\limits_1$, from 0 to M^{2m-1} in $\sum\limits_2$ and from $M^{2m+1} + 1$ to n in $\sum\limits_3$.

We write

$$A_m(\omega) = \sum_1 a_r X_r(\omega) x_m^r \quad \ldots(6)$$

and

$$R_m(\omega) = (\sum_2 + \sum_3) \; a_r X_r(\omega) \; x_m^r \, . \quad \ldots(7)$$

Obviously A_m and A_{m+1} are independent random variables.

Lemma 2.1—If $h(t)$ is a positive slowly varying function in the neighbourhood of the origin, then for $\rho > 0$,

(i) $\lim\limits_{t \to 0} \; t^{\rho} h(t) = 0$

(ii) $\lim\limits_{t \to 0} \; t^{-\rho} h(t) = \infty$.

These results follow from Karamata's theorem (cf. Ibragimov and Linnik[2], p. 795). We define normalizing constants V_m by

$$V_m^{\alpha} = \sum_1 |a_r|^{\alpha} \; x_m^{\alpha r} \; h(a_r \; x_m^r \; \theta/V_m)$$

where θ is an arbitrary small positive number.

Lemma 2.2—$V_m > t_n \left(\frac{d}{e} \; M^{2m} \right)^{1/\alpha}$ for $d > 0$.

PROOF : $h(t)$ may be bounded or unbounded when $t \to 0$. If $\lim\limits_{t \to 0} \; h(t) = \infty$, then there exists $t_1 > 0$ such that $h(t) > 1$, for $t < t_1$. If $h(t)$ is bounded, as $h(t)$ is positive in the neighbourhood of the origin, there exists $d > 0$ such that $h(t) > d$. Hence $h(t) > d$ for both the cases. So for large n, we get

$$d > \frac{1}{\log n \log \left(\left(\frac{k_n}{t_n} \right) (\log n)^{5/\alpha} \right)^2} . \quad \ldots(8)$$

Now

$$\begin{aligned} V_m^{\alpha} &= \sum_1 |a_r|^{\alpha} \; x_m^{\alpha r} \; h(a_r \; x_m^r \; \theta/V_m) \\ &> \sum_{M^{2m-1}+1}^{2m+1} d \; x_m^{\alpha r} \; t_n^{\alpha} \\ &> \sum_{M^{2m-1}+1}^{M^{2m}} d \; x_m^{\alpha r} \; t_n^{\alpha} \end{aligned}$$

(*equation continued on p.* 658)

$$= d\, t_n^{\alpha} \;(M^{2m} - M^{2m-1}) \left(1 - \frac{1}{M^{2m}} \right)^{M^{2m}}$$

$$> \frac{d}{e} t_n^{\alpha} \; M^{2m}.$$

Thus $V_m > t_n \left(\frac{d}{e} M^{2m} \right)^{1/\alpha}$. ...(9)

Lemma 2.3—Let

$$\Gamma_1 = \left\{ \omega : \left| \sum_3 a_r \, X_r(\omega) \, x_m^r \right| > \frac{V_m}{2} \right\}$$

$$T_2 = \left\{ \omega : \left| \sum_2 a_r \, X_r(\omega) \, x_m^r \right| > \frac{V_m}{2} \right\}$$

$$T = \{ \omega : | R_m(\omega) | > V_m \}$$

and G be the set of all points for $m = \left[\frac{k}{2} \right] + 1, \left[\frac{k}{2} \right] + 2, \dots k$ where $| R_m(\omega) | > V_m$.

Then

$$P(G) \leqslant 129\, e \; \frac{\log \left(\left(\frac{kn}{t_n} \right) (\log n)^{5/\alpha} \right)}{\log n}.$$

Proof : The characteristic function of $\sum_3 a_r \, X_r(\omega) \; x_m^r$ is given by

$$\phi_m(t) = \exp \{ - | t |^{\alpha} h_m(t) \}$$

where

$$h_m(t) = \sum_3 | a_r |^{\alpha} \, x_m^{\alpha r} \, h \left(a_r \, x_m^r \, t \right).$$

Now

$$P(T_1) < 2 - \left| \frac{V_m}{4} \int_{-4/V_m}^{4/V_m} \phi_m(t) \, dt \right|$$

(cf. Gnedenko and Kolmogorov, p. 54)

$$\leqslant \frac{V_m}{4} \int_{-4/V_m}^{4/V_m} | 1 - \phi_m(t) | \, dt.$$

But

$$h_m(t) = \sum_3 | a_r |^{\alpha} \, x_m^{\alpha r} \, h(a_r \; x_m^r \, t)$$

(*equation continued on p.* 659)

$$< | t |^{-\rho} \sum_{s} | a_r |^{\alpha-\rho} x_m^{r(\alpha-\rho)} .$$

Since by Lemma 2.1, for $\rho > 0$,

$$h(a_r x_m^r t) \leqslant | a_r x_m^r t |^{-\rho} \text{ as } t \to 0.$$

But

$$(1 - \phi m(t)) = 1 - \exp(- | t |^{\alpha} hm(t))$$

$$= | t |^{\alpha} hm(t) (1 + o(1)) \text{ as } t \to 0$$

$$< 2 | t |^{\alpha-\rho} \sum_{3} | a_r |^{\alpha-\rho} x_m^r (\alpha - \rho).$$

Now

$$P(T_1) \leqslant V_m \sum_{s} x_m^{r(\alpha-\rho)} | a_r |^{\alpha-\rho} \int_0^{4/V_m} t^{\alpha-\rho} dt$$

$$= \frac{2^{2\alpha-2\rho+2}}{\alpha-\rho+1} \frac{\sum_{s} x_m^{r(\alpha-\rho)} | a_r |^{\alpha-\rho}}{V_m^{\alpha-\rho}}$$

$$< \frac{2^{2\alpha-2\rho+2}}{\alpha-\rho+1} \frac{\sum_{s} x_m^{r(\alpha-\rho)} k_n^{\alpha-\rho}}{V_m^{\alpha-\rho}}.$$

But

$$\sum_{s} x_m^{r(\alpha-\rho)} \leqslant \frac{2\alpha}{\alpha - \rho} M^{2m-1} .$$

So

$$P(T_1) \leqslant 2^{2\alpha+3} \frac{\alpha}{(\alpha - \rho)(\alpha - \rho + 1)} \left(\frac{e}{d} \right)^{1-\rho/\alpha}$$

$$\times \left(\frac{k_n}{t_n} \right)^{\alpha-\rho} \frac{1}{\left(\frac{k_n}{t_n} \right)^{\alpha\left(1 - \frac{2m\rho}{\alpha}\right)} (\log n)^{5\left(1 - \frac{2m\rho}{\alpha}\right)}}$$

(by Lemma 2.2 and (4))

$$\leqslant \frac{512}{\alpha + 2} \left(\frac{e}{d} \right) \left(\frac{k_n}{t_n} \right)^{\alpha} \frac{1}{\left(\frac{k_n}{t_n} \right)^{(\alpha-2m\rho)} \log n^{\left(5 - \frac{10m\rho}{\alpha}\right)}}$$

(*equation continued on p. 660*)

$$\left(\because \frac{k_n}{t_n} > 1 \text{ for large } n\right)$$

$$\leqslant 171\left(\frac{e}{d}\right)\left(\frac{k_n}{t_n}\right)^{2m\rho} \frac{1}{(\log n)^{\left(c - \frac{10m\rho}{\alpha}\right)}}$$

$$\alpha \geqslant 1$$

$$\left(\because \frac{1}{\alpha+2} < \frac{1}{3}\right).$$

Choose $\rho = \dfrac{\alpha}{10m}$ for a fixed m and by this choice we have,

$$P(T_1) < 171\left(\frac{e}{d}\right)\frac{1}{(\log n)^3}. \qquad \ldots(10)$$

Then proceeding as in above we get

$$P(T_2) \leqslant \frac{2^{2\alpha-2\rho+2}}{\alpha-\rho+1} \frac{\sum_2 x_m^{r(\alpha-\rho)} \mid a_r \mid^{\alpha-\rho}}{V_m^{\alpha-\rho}}.$$

But

$$\sum_2 x_m^{r(\alpha-\rho)} \leqslant 1 + M^{2m-1} \leqslant 2M^{2m-1}.$$

So

$$P(T_2) \leqslant 86\left(\frac{e}{d}\right)\frac{1}{(\log n)^3}. \qquad \ldots(11)$$

Now

$$\mid R_m \mid = \left| \sum_2 a_r X_r(\omega) x_m^r + \sum_3 a^r X_r(\omega) x_m^r \right|$$

$$\leqslant \left|\sum_2 a_r X_r(\omega) x_m^r\right| + \left|\sum_3 a_r X_r(\omega) x_m^r\right|$$

$$\leqslant \frac{V_m}{2} + \frac{V_m}{2} = V_m.$$

Obviously,

$$P(T) \leqslant P(T_1) + P(T_2)$$

$$\leqslant 257\left(\frac{e}{d}\right)\frac{1}{(\log n)^3}$$

(by (10) and (11)).

Then

$$P(G) \leqslant \sum_{m=1}^{k} P(T) = k\,P(T)$$

$$\leqslant 257 \left(\frac{e}{d} \right) \frac{k}{(\log n)^3}$$

$$< 129\, e \, \frac{\log \left(\left(\frac{k_n}{t_n} \right) (\log n)^{5/\alpha} \right)}{\log n} \qquad \ldots (12)$$

(by (4) and (8)).

§ 3. Proof of the Theorem—Let

$$E_m = \{\omega\colon A_{2m} \geqslant V_{2m},\ A_{2m+1} < -V_{2m+1}\}$$

$$F_m = \{\omega\colon A_{2m} < -V_{2m},\ A_{2m+1} \geqslant V_{2m+1}\}$$

$$D_1 = \{\omega\colon A_{2m} \geqslant V_{2m},\ |R_{2m}| < V_{2m}\}$$

$$D_2 = \{\omega\colon A_{2m+1} < -V_{2m+1},\ |R_{2m+1}| < V_{2m+1}\}$$

$$D_3 = \{\omega\colon A_{2m} < -V_{2m},\ |R_{2m}| < V_{2m}\}$$

$$D_4 = \{\omega\colon A_{2m+1} \geqslant V_{2m+1},\ |R_{2m+1}| < V_{2m+1}\}$$

and

$$H_1 = D_1 \cap D_2,\ H_2 = D_3 \cap D_4.$$

Obviously when H_1 occurs, $f(x_{2m}) > 0$ and $f(x_{2m+1}) < 0$, and also when H_2 occurs, $f(x_{2m}) < 0$ and $f(x_{2m+1}) > 0$.

Therefore, if $H_1 \cup H_2$ occurs, $f(x)$ has a root in (x_{2m}, x_{2m+1}). Again we are to show that $P(E_m \cup F_m) > 0$.

Let $G_m(x)$ and $g_m(t)$ be respectively the distribution function and the characteristic function of (A_m/V_m). Then

$$P(E_m \cup F_m) = (1 - G_m(1))\, G_{2m+1}(-1) + G_{2m}(1)\,(1 - G_{2m+1}(1)). \qquad \ldots (13)$$

Also

$$g_m(t) = \exp\left\{ -|t|^{\alpha} \frac{1}{V_m^{\alpha}} \sum_{1} a_r\, x_m^{\alpha r}\, h\left(x_m^{r}\, t/V_m \right) \right\}$$

$$= \exp\left\{ -|t|^{\alpha} \frac{1}{V_m^{\alpha}} \left| \frac{\theta}{t} \right|^{o(1)} \sum_{1} a_r\, x_m^{\alpha r}\, h\left(x_m^{r}\, \theta/V_m \right) \right.$$

(*equation continued on p. 662*)

$$(1 + o(1)) \Big\} \text{ (by lemma (2.1))}$$

$$= \exp\{ | - t |^{\alpha - o(1)} \, \theta^{o(1)} (1 + o(1))\}$$

(by the definition of V_m).

Therefore as $m \to \infty$, $g_m(t) \to \exp(- | t |^{\alpha})$ in any bounded interval of t-values.

Hence

$$\sup_x | G_m(x) - F(x) | = o(1). \qquad \text{...(14)}$$

So for $\epsilon > 0$, we have

$$| G_{2m}(-1) - F(-1) | < \epsilon$$

and

$$| G_{2m+1}(-1) - F(-1) | < \epsilon.$$

So

$$G_{2m}(-1) > F(-1) - \epsilon, \quad G_{2m+1}(-1) > F(-1) - \epsilon$$

and

$$1 - G_{2m}(1) > 1 - F(1) - \epsilon, \; 1 - G_{2m+1}(1) > 1 - F(1) - \epsilon.$$

Hence

$$P(E_m \cup F_m) \geqslant 2(F(-1) - \epsilon)(1 - F(1) - \epsilon)$$

$$\to 2F(-1)(1 - F(1) \text{ as } \epsilon \to 0 \text{ when } m \to \infty \text{ with } n.$$

Thus

$$P(E_m \cup F_m) = \delta_m = \delta > 0. \qquad \text{...(15)}$$

Where δ is an absolute constant.

Let η_m be the indicator function of the event $E_m \cup F_m$.

Then by (15)

$$P\{\omega: \eta_m = 1\} = \delta_m \text{ and } P\{\omega: \eta_m = 0\} = 1 - \delta_m.$$

Thus η_m's are independent random variables with

$$E(\eta_m) = \delta_m \text{ and } V(\eta_m) = (\delta_m - \delta_m^2).$$

It follows from Samal[4] (p. 439) that there are at least

$$\tfrac{1}{2}\left\{k - \left[\frac{k}{2}\right] - 2\right\} \text{ pairs } (A_{2p}, A_{2p+1}) \text{ such that } \left[\frac{k}{2}\right] + 1 \leqslant 2p \leqslant 2p + 1 \leqslant k.$$

If q be the number of such pairs, then

$$q \geqslant \tfrac{1}{2}\left(k - \left[\frac{k}{2} \right] - 2 \right)$$

$$\leqslant \tfrac{1}{8}k \text{ (for large } k)$$

$$\leqslant \frac{\log n}{32 \log \left(\left(\frac{kn}{tn} \right) (\log n)^{5/\alpha} \right)} \text{ (by (4)).} \qquad \ldots(16)$$

Let $\eta = \sum_q \eta m$, then for $0 < \epsilon_1 < \delta_m$, by Chebysheff's inequality

$$P \mid \eta - E(\eta) \mid \geqslant q\,\epsilon_1 \leqslant \frac{V(\eta)}{q^2 \epsilon_1^2} \leqslant \frac{1}{q\,\epsilon_1^2}$$

for large n, take

$$\epsilon_1 > \frac{1}{\left\{ \log \left(\left(\frac{kn}{tn} \right) \log n)^{5/\alpha} \right) \right\}^{1/4}} .$$

Then we have,

$$P \{ \mid \eta - E(\eta) \mid \geqslant q\,\epsilon_1 \} \leqslant \frac{32 \left\{ \log \left(\left(\frac{kn}{tn} \right) (\log n)^{5/\alpha} \right) \right\}^{3/2}}{\log n}$$

(by (16)).

Thus

$\mid \eta - E(\eta) \mid < q\,\epsilon_1$ on a set S_1 for which

$$P(S_1) \leqslant \frac{32\, e \left\{ \log \left(\left(\frac{kn}{tn} \right) (\log n)^{3/\alpha} \right) \right\}^{3/2}}{\log n} . \qquad \ldots(17)$$

Again $\eta \geqslant E(\eta) - q\,\epsilon_1 \geqslant q\,(\theta - \epsilon_1)$.

Taking $\delta > 2\,\epsilon_1$ for large n, we have

$$\eta > q\,\epsilon_1 > \frac{\log n}{32 \left\{ \log \left(\left(\frac{kn}{tn} \right) (\log n)^{5/\alpha} \right) \right\}^{5/4}}$$

$$> \frac{\log n}{32 \log \left(\left(\frac{kn}{tn} \right) (\log n)^{5/\alpha} \right)} .$$

Hence

$$N_n \geqslant \frac{\log n}{32 \log \left(\left(\frac{kn}{tn} \right) (\log n)^{5/\alpha} \right)} . \qquad \ldots(18)$$

Now if S denotes the entire exceptional set

Then

$$P(S) \leqslant P(G) + P(S_1)$$

$$< \frac{161\, e \log \left(\left(\frac{k_n}{t_n} \right) (\log n)^{5/\alpha} \right)}{\log n}. \quad \text{.. (19)}$$

Therefore from (18) and (19), we have

$$P\left\{\omega: N_n(w) < \frac{\log n}{32 \log \left(\left(\frac{k_n}{t_n} \right) (\log n)^{5/\alpha} \right)} \right\}$$

$$< \frac{161\, e \log \left(\left(\frac{k_n}{t_n} \right) (\log n)^{5/\alpha} \right)}{\log n}.$$

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REFERENCES

1. B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions of Sums of Independent Random Variables.* Addison Wesley, Inc., 1968, p. 171.
2. I. A. Ibragimov and Yu. V. Linnik. *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff, Groningen, 1972.
3. J. E. A. Dunnage, *Proc. London Math. Soc.* **18** (3) (1968), 439–60.
4. G. Samal, *Proc. Camb. Phil. Soc.* **58** (1962), 433–42.
5. G. Samal and M. N. Mishra, *Proc. Amer. Math. Soc.* **39** (1973), 184–89.
6. M. N. Mishra, N. N. Nayak and S. Pattanayak, *J. Aust. Math. Soc.* (A) **35** (1983), 18–27.

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GRONWALL, BIHARI AND LANGENHOP TYPE INEQUALITIES FOR DISCRETE PFAFFIAN EQUATION

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The object of this paper is to establish some new discrete inequalities of the Gronwall, Bihari and Langenhop type for Pfaffian equation.

INTRODUCTION

It is well known that the discrete analogue of celebrated Gronwall's lemma[4] and its generalization, Bihari's lemma[3] established by Jones[5] and Sugiyama[9] (also see Agarwal and Thandapani[1,2] and Pachpatte[8]) play an important role in the theorey of difference equations and numerical analysis. They have been used, for example, to obtain upper bounds of solutions as in Langenhop[7] it is shown that lower bounds can be obtained similarly. The purpose of this note is to demonstrate how similar estimates can be derived for solutions of discrete Pfaffian equations and for the continuous case similar results are available in Grudo and Yarchuk[6].

Before giving the main result, we shall first introduce some notations which we shall use throughout the paper. N denotes the set $\{0, 1, \ldots\}$. The expression $\sum_{s=0}^{t-1} b(s)$ represents a solution of the linear difference equation $\Delta z(t) = b(t)$ for all $t \in N$ under the initial condition $z(0) = 0$ where Δ is the operator defined by $\Delta z(t) = z(t+1) - z(t)$. It is supposed that $\sum_{s=0}^{-1} b(s) = 0$. The expression $\prod_{s=0}^{t-1} c(s)$ represents a solution of the linear difference equation $z(t+1) = c(t)\, z(t)$ for all $t \in N$ with the initial condition $z(0) = 1$. It is supposed that $\prod_{s=0}^{-1} c(s) = 1$. We also define

$$\Delta z_{t_1}(t_1, t_2) = z(t_1 + 1, t_2) - z(t_1, t_2)$$

$\Delta z_{t_2}(t_1, t_2) = z(t_1, t_2 + 1) - z(t_1, t_2)$ where the letters t_1 and t_2 are used to denote the two independent variable which are members of N.

MAIN RESULT

Consider the discrete Pfaffian equation in two independent variables :

$$x(t_1, t_2) = x(0,0) + \sum_{s_1=0}^{t_1-1} f_1(s_1, t_2, x(s_1, t_2)) + \sum_{s_2=0}^{t_2-1} f_2(0, s_2, x(0, s_2)) \quad \ldots(1)$$

where

f_j $(j = 1, 2)$ are real-valued and defined for all $t_1 \geqslant 0$, $t_2 \geqslant 0$

and

$$| f_j(t_1, t_2, x) | \leqslant F_j(t_1, t_2) W_j(|x|) \; (j = 1, 2) \quad \ldots(2)$$

where the F_j are real-valued, non negative and defined for all $t_1 \geqslant 0$, $t_2 \geqslant 0$ and $W_j(u)$ are continuous and non decreasing for $u \geqslant 0$ and $W_j(u) > 0$ for $u > 0$.

Theorem —Let $| x(t_1, t_2) | > 0$ for $t_1 \geqslant 0$ and $t_2 \geqslant 0$. Then, if $t_1 \geqslant 0$, $t_2 \geqslant 0$, we have

$$| x(t_1, t_2) | \leqslant G_1^{-1}\left\{\sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) + G_1\left[G_2^{-1}\left(\sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right)\right]\right\} \quad \ldots(3)$$

where,

$$G_i(u) = \int_{u_0}^{u} \frac{d\,u_1}{W_i(u_i)} \; (0 < u < \bar{u}_i \,,\; u_0 = | x(0,0) |) \quad \ldots(4)$$

and

G_i^{-1} $(i = 1, 2)$ are the inverses of G_i.

Furthermore,

$$| x(t_1, t_2) | \geqslant G_1^{-1}\left\{- \sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) + G_1\left[G_2^{-1}\left(- \sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right)\right]\right\} \quad \ldots(5)$$

for all $t_1 \geqslant 0$, $t_2 \geqslant 0$ for which

$$- \sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) + G_1\left[G_2^{-1}\left(- \sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right)\right] \in \operatorname{dom}(G_1^{-1})$$

and

$$- \sum_{s_2=0}^{t_2-1} F_2(o, s_2) \in \operatorname{dom}(G_2^{-1}).$$

PROOF : By putting $|x(t_1, t_2)| = u(t_1, t_2)$, we conclude from (2) that

$$u(t_1, t_2) \leqslant u_0 + \sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) W_1(u(s_1, t_2)) + \sum_{s_2=0}^{t_2-1} F_2(0, s_2) W_2(u(0, s_2)). \quad \ldots(6)$$

Now put

$$R(t_1, t_2) = u_0 + \sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) W_1(u(s_1, t_2)) + \sum_{s_2=0}^{t_2-1} F_2(0, s_2) W_2(u(0, s_2))$$

to obtain

$$\Delta R_{t_1} = F_1(t_1, t_2) W_1(u(t_1, t_2)) \leqslant F_1(t_1, t_2) W_1(R(t_1, t_2))$$

since $W_1(u)$ is non decreasing and $u(t_1, t_2) \leqslant R(t_1, t_2)$. Since $R(t_1, t_2) > 0$ and $W_1(u) > 0$ for $u > 0$ for $u > 0$, this inequality together with (4) yields

$$G_1(R(t_1, t_2)) - G_1(R(0, t_2)) \leqslant \sum_{s_1=0}^{t_1-1} F_1(s_1, t_2). \quad \ldots(7)$$

From (6), we have, on putting $t_1 = 0$

$$u(0, t_2) \leqslant u_0 + \sum_{s_2=0}^{t_2-1} F_2(0, s_2) W_2(u(0, s_2)).$$

As before, we conclude that

$$G_2(R(0, t_2)) - G_2(R(0, 0)) \leqslant \sum_{s_2=0}^{t2-1} F_2(0, s_2).$$

Since $R(0,0) = u_0$ and $G_2(u_0) = 0$, we have

$$R(0, t_2) \leqslant G_2^{-1}\left[\sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right]. \quad \ldots(8)$$

Hence, since $G_1(R(t_1, t_2)) \geqslant G_1(u(t_1, t_2))$, we conclude from (7) and (8) that

$$G_1(u(t_1, t_2)) \leqslant G_1\left\{G_2^{-1}\left[\sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right]\right\} + \sum_{s_1=0}^{t_1-1} F_1(s_1, t_2)$$

and the result (3) follows.

We now prove (5). Again using the notation $|x(t_1, t_2)| = u(t_1, t_2)$, We find from (1) and (2)

$$u(t_1, t_2) \geqslant u(s_1, s_2) - \sum_{r_1=s_1}^{t_1-1} F_1(r_1, t_2) W_1(u(r_1, t_2)$$

$$- \sum_{r_2=s_2}^{t_2-1} F_2(s_1, r_2) W_2(u(s_1, r_2)) \quad \text{...(9)}$$

for $t_1 > 0$, $t_2 > 0$ and $0 \leqslant s_i \leqslant t_i$ $(i = 1, 2)$.

Let

$$R(s_1, s_2) = u(t_1, t_2) + \sum_{r_1=s_1}^{t_1-1} F_1(r_1, t_2) W_1(u(r_1, t_2))$$

$$+ \sum_{r_2=s_2}^{t_2-1} F_2(s_1, r_2) W_2(u(s_1, r_2)).$$

Then

$$\Delta R_{s_2} = - F_2(s_1, s_2) W_2(u(s_1, s_2))$$

and since $R(s_1, s_2) \geqslant u(s_1, s_2)$

$$\Delta R_{s_2} \geqslant - F_2(s_1, s_2) W_2(R(s_1, s_2)).$$

Thus

$$G_2(R(s_1, t_2)) - G_2(R(s_1, s_2)) \geqslant - \sum_{r_2=s_2}^{t_2-1} F_2(s_1, r_2)$$

and we conclude that

$$G_2(R(s_1, t_2)) \geqslant G_2(u(s_1, s_2)) - \sum_{r_2=s_2}^{t_2-1} F_2(s_1, r_2).$$

This implies that

$$R(s_1, t_2) \geqslant G_2^{-1}[G_2(u\, s_1, s_2)) - \sum_{r_2=s_2}^{t_2-1} F_2(s_1, r_2)]. \quad \text{...(10)}$$

Putting $s_2 = t_2$ in (9), we obtain

$$u(t_1, t_2) \geqslant u(s_1, t_2) - \sum_{r_1=s_1}^{t_1-1} F_1(r_1, t_2) W_1(u(r_1, t_2)).$$

Hence

$$G_1(R(t_1, t_2)) - G_1(R(s_1, t_2)) \geqslant - \sum_{r_1=s_1}^{t_1-1} F_1(r_1, t_2).$$

Since $R(t_1, t_2) = u(t_1, t_2)$, (10) implies that

$$G_1(u(t_1, t_2)) \geqslant -\sum_{r_1=s_1}^{t_1-1} F_1(r_1, t_2) + G_1\left\{ G_2^{-1}\left[G_2(u(s_1, s_2)) - \sum_{r_2=s_2}^{t_2-1} F_2(s_1, r_2)\right]\right\}$$

and putting $s_1 = 0$, $s_2 = 0$, we obtain

$$G_1(u(t_1, t_2)) \geqslant -\sum_{r_1=0}^{t_1-1} F_1(r_1, t_2) + G_1\left\{ G_2^{-1}\left[-\sum_{r_2=0}^{t_2-1} F_2(0, r_2)\right]\right\}.$$

This proves (5).

Remark 1: If $W_1(u) = W_2(u)$, (3) and (5) simplify :

$$|x(t_1, t_2)| \leqslant G_1^{-1}\left[\sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) + \sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right]$$

$$|x(t_1, t_2)| \geqslant G_1^{-1}\left[\sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) - \sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right].$$

Remarks 2 : If $W_1(u) = W_2(u) = u$, we have

$$|x(t_1, t_2)| \leqslant |x(0, 0)| \prod_{s_1=0}^{t_1-1} [1 + F_1(s_1, t_2)] \prod_{s_2=0}^{t_2-1} [1 + F_2(0, s_2)]$$

$$\leqslant |x(0, 0)| \exp\left[\sum_{s_1=0}^{t_1-1} F_1(s_1\, t_2) + \sum_{s_2=0}^{t_1-1} F_2(0, s_2)\right].$$

$$|x(t_1, t_2)| \geqslant |x(0, 0)| \prod_{s_1=0}^{t_1-1} [1 - F_1(s_1, t_2)] \prod^{t_2-1} [1 - F_2(0, s_2])$$

$$\geqslant |x(0, 0)| \exp\left[-\sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) - \sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right].$$

References

1. R. P. Agarwal and E. Thandapani, *Appl. Math. Comp.* **7** (1980), 205–24.
2. R. P. Agarwal and E. Thandapani, *Bull. Inst. Math. Acad Sinica* **9** (1981), 235–48.
3. I. Bihari, *Acta Math. Acad. Sci. Hungar.* **8** (1957), 261–78.
4. T. H. Gronwall, *Ann. Math.* **2** (1919), 292-96.
5. G. S. Jones, *STAM J Appl. Math.* **12** (1964), 43–57.

6. E. I. Grudo and L. F. Yarchuk, *Differentsial'nye Uraveniya* **13** (1977), 749–52.
7. C. E. Langenhop, *Proc. Am. Math. Soc.* **11** (1960), 796–99.
8. B. G. Pachpatte, *J. Indian Math. Soc.* **37** (1973), 147–56.
9. S. Sugiayama, *Bull. Sci. Engr. Research Lab. Wasada Univ.* **45** (1969), 140–44.

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A NOTE ON PRIMARY DECOMPOSITION IN NOETHERIAN NEAR-RINGS

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Hans H. Storrer[3] has introduced the notion of critical left ideals in rings and has given a generalisation of primary decomposition to non-commutative rings. In this paper critcal left ideals in Near-rings are introduced and the relation between critical left ideals and left ideals of type O is studied. Further primary decomposition is also exhibited in modules over near-ring under some conditions.

1. INTRODUCTION

All near-rings are assumed to be zero-symmetric right near-rings with identity. Throughout this paper near-ring under consideration is denoted by K. The definitions of K-module, K-subgroups and submodules are as given in Pilz[2]. However for the sake of continuity the definitions are given again.

Definition 1.1—Let $(M, +)$ be a group and K be a near-ring such that there exists a mapping $u : K \times M \to M$ satisfying the conditions.

$$(k + k')\, m = k\, m + k'\, m$$

$$(kk')\, m = k\, (k'\, m).$$

$1.m = m$ for all $k, k' \in K, m \in M$ and 1 is the identify of K. Then $(M, +, u)$ is called a K-module.

Definition 1.2—A subgroup N of K-module M is said to be a K-subgroup of M if $(N, +)$ is a subgroup with $KN \subseteq N$.

Definition 1.3—A normal subgroup N of M is called a submodule of M if $k\,(m + n) - k\, m \in N$. For all $m \in M, n \in N$ and $k \in K$.

The concepts of essential extension and rational extension were introduced by Barua[1] for near-rings. Here we give a variant of the above notions similar to modules over a ring.

Definition 1.4—Let M be a module over a near-ring K. M is said to be (a) a 'module essential extension' of a non-zero K-subgroup N if for every non-zero submodule N' of M, $N \cap N' \neq 0$; (b) an 'essential extension' of a non-zero K-subgroup N if for every non-zero k-subgroup N', $N \cap N' \neq 0$.

Clearly essential extension implies module essential extension. The converse need not be true.

Example—Let $K = \{0, a, b, c\}$ be the klien group under addition and the multiplication be defined by the following table.

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then $M = K$ is module essential extension of $N = \{0, a\}$ but not an essential extension of N.

Definition 1.5 : *Rational Extension*—Let N be a non-zero K-subgroup of a K-module M. M is said to be a rational extension of N if the following condition is satisfied :

$N \subsetneq A \subsetneq M$, A is a K-subgroup of M and $f : A \to M$ is a

K-homomorphism such that $\ker f \supset N$ implies $f = 0$.

In modules over rings it is known that a rational extension implies essential extension. This need not be the case in case of near-ring modules as seen from the following example.

Example : The dihedral group $D_8 = \{0, a, 2a, 3a, b, a + b, 2a + b, 3a + b\}$. With the addition and multiplication operations defined below is a near-ring with identity a. It is a rational extension of each of its non-zero K-subgroups but not an essential extension of its non-zero K-subgroups.

$+$	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
0	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
a	a	$2a$	$3a$	0	$a+b$	$2a+b$	$3a+b$	b
$2a$	$2a$	$3a$	0	a	$2a+b$	$3a+b$	b	$a+b$
$3a$	$3a$	0	a	$2a$	$3a+b$	b	$a+b$	$2a+b$
b	b	$3a+b$	$2a+b$	$a+b$	0	$3a$	$2a$	a
$a+b$	$a+b$	b	$3a+b$	$2a+b$	a	0	$3a$	$2a$
$2a+b$	$2a+b$	$a+b$	b	$3a+b$	$2a$	a	0	$3a$
$3a+b$	$3a+b$	$2a+b$	$a+b$	b	$3a$	$2a$	[illegible]	0

	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
0	0	0	0	0	0	0	0	0
a	0	a	$2a$	$3a$	b	$a+b$	$2a+b$	$3a+b$
$2a$	0	$2a$	0	$2a$	0	0	0	0
$3a$	0	$3a$	$2a$	a	b	$a+b$	$2b+b$	$3a+b$
b	0	b	$2a$	b	b	$a+b$	$2a+b$	$3a+b$
$a+b$	0	$a+b$	0	$3a+b$	0	0	0	0
$2a+b$	0	$2a+b$	$2a$	$2a+b$	b	$a+b$	$2a+b$	$3a+b$
$3a+b$	0	$3a+b$	0	$a+b$	0	0	0	0

Definition 1.6—A K-module M is said to be (i) 'Uniform' if it is an essential extension of each of its non-zero K-subgroups; and

(ii) 'Strongly uniform' if it is uniform and is a rational extension of each of its non-zero K-subgroups.

Notation : If X is a subset of a K-module M then $<X>$ stands for the submodule of M generated by X.

We assume that K-module M satisfies the property (p) :

"$< N_1 \cap N_2 > = < N_1 > \cap < N_2 >$ for any two K-subgroups N_1 and N_2 of M".

Any near-ring K in which every K-subgroup is a submodule of K satisfies this property.

Proposition 1.7—Let M be a K-module with ascending chain condition (A.C.C.) on K-subgroups and satisfying the property (p). Then M has a submodule which is uniform.

Proof : Suppose M has no submodule which is uniform. Then there exists non-zero K-subgroups N_1 and N_1' such that $N_1 \cap N_1' = 0$. Therefore $M_1 \cap M_1' = 0$ where $M_1 = < N_1 >$ and $M_1' = < N_1' >$. By hypothesis M_1 is not uniform. So there exists non-zero K-subgroups N_2 and N_2' of M_1 such that $N_2 \cap N_2' = 0$. Therefore $M_2 \cap M_2' = 0$ where $< N_2 > = M_2$ and $< N_2' > = M_2'$. Repeating the argument, we get a sequence $\{M_n\}$ of submodules of M which are not uniform such that $M_1 \subset M_1 + M_2 \subset M_1 + M_2 + M_3 \subset \ldots$ contradicting the A.C.C. on K-subgroups.

Hence M has a submodule which is uniform.

2. Critical Left Ideals

Following the notion of critical left ideal as given by Storrer[3] the concept of critical left ideals in near-rings is introduced.

Definition 2.1—A left ideal P of a near-ring K is called a critical left ideal if $M = K/P$ is a strongly uniform K-module.

Definition 2.2—(*a*) A left ideal L of K is of type 2 if $M = K/L$ is a K-module which has no nontrivial K-subgroups.

(b) A left ideal L of K is of type O if $M = K/L$ is a K-module which has no nontrivial submodules.

From the definition it follows that every left ideal of type 2 is a critical left ideal. However we can still strengthen the result as follows.

Proposition 2.3—Every left ideal of type O of K is a critical left ideal.

PROOF : Let L be any left ideal of type O of K. Suppose $\Delta_i = L_i/L$, $i = 1, 2$ are two non-zero K-subgroups of $M = K/L$. Assume that $L_1 \not\subset L_2$ and $L_2 \not\subset L_1$. Choose $r_1 \in L_1 \setminus L_2$ and $r_2 \in L_2 \setminus L_1$. Since L is a modular left ideal of type O, $(K : L) = P$ is a O-promitive ideal of K and $M = K/L$ is a K/P-module of type O.

By density theorem (Pilz[2], 115) there exists $n + p \in N/P$ such that $(n + p)(r_2 + L) = r_1 + L$ which implies that $nr_2 - r_1 \in L \subset L_1$. Consequently $nr_2 \in L_1 \cap L_2$ and $nr_2 \notin L$. Thus $nr_2 + L \in \Delta_1 \cap \Delta_2$. This shows that intersection of any two non-zero K-subgroups of K/L is non-zero. That is $M = K/L$ is uniform.

Now to establish that $M = M/L$ is a rational extension of each of its non-zero K-subgroups, assume that there exists two non-zero K-subgroups Δ_1 and Δ_2 of M such that $\Delta_1 \subset \Delta_2$ and a K-homomorphism $f : \Delta_2 \to M$ with $\ker f \supset \Delta_1$. As before suppose $\Delta_i = L_i/L$, $i = 1, 2$. If $L_1 = L_2$, then clearly $f = 0$. Otherwise choose any $r_2 \in L_2 \setminus L_1$ and $r_1 \in L_1 \setminus L$. Considering $M = K/L$ as a K/P-module and appealing to density theorem again, there exists $n + p \in N/P$ such that $(n + p)(r_1 + L) = r_2 + L$ which implies $nr_1 - r_2 \in L$. Since f is a K-homomorphism and $nr_1 + L \in \Delta_1$ we have $f(r_2 + L) = f(nr_1 + L) = 0$.

This implies that $f = 0$ and M in a rational extension of Δ_1. Hence L is a critical left ideal of K.

In the class of all critical left ideals of a near-ring K, we introduce a relation as follows.

Definition 2.4—If I and J are two left ideals of K, I is said to be related to J if there exists $a \notin I$, $b \notin J$ such that

$$Ia^{-1} = Jb^{-1} \text{ where } Ia^{-1} = \{r \in K/ra \in I\} \text{ and }$$
$$Ja^{-1} = \{r \in K/rb \in J\}.$$

Lemma 2.5—Let K be a near-ring and P and Q be two critical left ideals of K then the following statements are equivalent :

(i) P is related to Q.

(ii) A non-zero K-subgroup of K/P is isomorphic to a non-zero K-subgroup of K/Q.

PROOF : Suppose P and Q are related, then there exists a and b such that

$$Pa^{-1} = Qb^{-1} \text{ and } a \notin P, b \notin Q.$$

Then Pa^{-1} is a left ideal of K and K/Pa^{-1} is a non-zero K-module Define :

$$\psi : K/Pa^{-1} \to K/P$$

by

$$\psi(x + Pa^{-1}) = xa + P.$$

The map ψ is an isomorphism of K/Pa^{-1} onto a K-subgroup L/P of K/P. Similarly K/Qb^{-1} is isomorphic to a K-subgroup L'/Q of K/Q showing that $L/P \simeq L'/Q$ if $Pa^{-1} = Qb^{-1}$.

Conversely suppose a non-zero K-subgroup N/P is isomorphic to a non-zero K-subgroup M/Q of K/Q.

Let ϕ be a K-isomorphism from N/P to M/Q.

Since $\phi \neq 0$, there exists $\phi(a + P) = b + Q \neq 0$.

$$\begin{aligned}\phi[r(a + p)] &= \phi(ra + P)\\ &= r\phi(a + P)\\ &= r(b + Q) = rb + Q.\end{aligned}$$

Since ϕ is mono, $ra \in P \Leftrightarrow rb \in Q$ that is $Pa^{-1} = Qb^{-1}$.

Proposition 2.6—The relation P is related to Q is an equivalence relation in the class of all critical left ideals.

PROOF : The only condition to be verified is the following : If I, J, L are critical left ideals such that I is related to J and J is related to L then I is related to L.

By Lemma 2.4, there exists non-zero K-subgroups M_1/I, N_1/J of K/I and K/J respectively such that $M_1/I \simeq N_1/J$. Similarly there exists non-zero K-subgroups N_2/J and M_2/L of K/J and K/L respectively such that $N_2/J \simeq M_2/L$. Since K/J is uniform, we have $N_1 \cap N_2 \underset{\neq}{\supset} J$.

Put $N = N_1 \cap N_2$. Then N/J is isomorphic to a non-zero K-subgroups of K/I and also N/J is isomorphic to a non-zero K-subgroup of K/L. Hence a non-zero K-subgroup of K/I is isomorphic to non-zero K-subgroup of K/L. Hence I is related to L.

The equivalence class containing P is denoted by $[P]$

3. Associated Left Ideals of a Module M

Definition 3.1—A critical left ideal P is said to belong to M if there exists $0 \neq x \in M$ such that Ann $(x) = P$.

Here we identify, the critical left ideals related to P and say $[P]$ is associated to M. With this identification, the set of all critical ideals belonging to M is denoted by Ass M.

Theorem 3.2—If M satisfies A.C.C. on K-subgroups and also satisfies property (p). Then there exists a non-zero K-subgroup B of M which is strongly uniform.

Proof : By Proposition 1.7, there exists a non-zero K-subgroup N of M which is uniform.

If N is strongly uniform, nothing to prove. Suppose now N is not strongly uniform. But by our choice N is uniform.

Therefore N cannot be a rational extension of each of its non-zero K-subgroups.

That is, there exists a maximal K-subgroup A of N such that N is not a rational extension of A (since M satisfies A.C.C. on K-subgroups). Hence there exists K-subgroup N' such that $A \underset{\neq}{\subset} N' \subset N$, and $\phi : N' \to N$, a non-zero homomorphism with Ker $\phi \supset A$.

Let $\phi(N') = B \neq 0$.

Then B is a non-zero K-subgroup of N. We claim that B is a rational extension of each of its non-zero K-subgroups.

Let $B' \neq 0$ be a K-subgroup of B.

Consider $\phi^{-1}(B') = A'$ then $A \underset{\neq}{\subset} A' \subset N$ and N is a rational extension of A' (by the maximality of A).

Let $f : B'' \to B$ be a homomorphism where $B' \subset B'' \subset B$ with Ker $f \supset B'$.

If $\phi^{-1}(B'') = A''$ then $\phi^{-1}(B) \supset \phi^{-1}(B'') \supset \phi^{-1}(B') \supset A'$.

Then $f \circ \phi : A'' \to N$ is a homomorphism such that Ker $f \circ \phi \supset A'$. But N is a rational extension of A'. Therefore $f \circ \phi = 0$ and hence $f = \circ$ on B''.

That is, B is a rational extension of B'.

Therefore $B \subset N$ and B is rational extension of each of its K-subgroup. As N is uniform, B is also uniform. Therefore B is strongly uniform.

As a corrollory to the above, we have :

Corollary 3.3—If M satisfies A C.C. on K-subgroups and also property (p), then Ass $M \neq \Phi$.

PROOF: By Theorem 3.2, M has a K-subgroup N which is strongly uniform. Let $0 \neq x \in N$. Then $Kx \simeq K/P$ where $P = \text{Ann}(x)$ and Kx is strongly uniform. Hence P is a critical left ideal associated to M and $\text{Ass}(M) \neq \Phi$. Some of the properties are the following :

Proposition 3.4—Let M be a K-module satisfying A.C.C. on K-subgroups.

(a) If M is the union of submodules of Mi, then $\text{Ass } M = U_i \text{ Ass } M_i$.

(b) If P is a critical left ideal then $\text{Ass}(K/P) = \{[P]\}$.

(c) If $N \subseteq M$, then $\text{Ass } N \subseteq \text{Ass } M \subseteq \text{Ass } N \cup \text{Ass } M/N$.

(d) If M is the direct sum of submodules M_i, then $\text{Ass } M = U_i \text{ Ass } M_i$

PROOF: (b) To show $\text{Ass}(K/P) = \{[P]\}$.

Consider $\text{Ann}(1 + P) = [r \in K / r(1 + P) = \overline{0}) = P$.

If there exists $x \in K$ such that $\text{Ann}(x + P) = Q$.

$$Q.1^{-1} = \text{Ann}(x + P) = [r \in K / r(x + P) = 0] = Px^{-1}.$$

Hence P is related to Q.

Therefore $\text{Ass}(K/P) = \{[P]\}$.

(c) If $N \subseteq M$, then $\text{Ass}(N) \subseteq \text{Ass}(M)$ is clear.

Let N be a submodule of M.

Let $P \in \text{Ass } M$ and $P = \text{Ann}(x)$, $0 \neq x \in M$.

Case (i)—If $Kx \cap N = <0>$

$$\begin{aligned} \text{Ann}(x + N) \text{ in } M/N &= [y \in K / y(x + N) = 0 + N] \\ &= [y \in K / yx = 0] \\ &= \text{Ann}(x) \text{ in } M. \end{aligned}$$

That is $P \in \text{Ass}(M/N)$.

Case (ii)—If $Kx \cap N \neq <0>$. Let $N' = Kx \cap N$ consider N' as K-module. Then $\text{Ass}(N') \neq \Phi$ by corollory 3.3.

Let $Q \in \text{Ass}(N') \subset \text{Ass } N$.

$Q \in \text{Ass}(N') \Rightarrow$ There exists $y \in N'$ such that $\text{Ann}(y) = Q$

$y \in Kx \Rightarrow$ There exists $a \in K$ such that $y = ax$.

$$\begin{aligned} Q = \text{Ann}(y) &= \{z / zy = 0\} \\ &= \{z / (za)x = 0\} \\ &= \{z / za \in P\} = Pa^{-1}. \end{aligned}$$

So $Pa^{-1} = Q1^{-1}$ and P is related to Q.

Hence $[P] \in \text{Ass}\,(N)$.

4. Primary Submodules

A K-module M is said to be coprimary if Ass M consists of a single element and and a submodule N of M is said to be primary if quotient module M/N is coprimary.

Theorem 4.1—Suppose M is a module which satisfies A.C.C. on K-subgroups and uniform. Then M is coprimary.

Proof : Let $P \in \text{Ass}\, M$, $P = \text{Ann}\,(x)$, $Kx \simeq K/P$.

If $Q \in \text{Ass}\, M$, then $Q = \text{Ann}\,(y)$, $Ky \simeq K/Q$.

Since M is uniform, $Kx \cap Ky \neq 0$.

Let $C = Kx \cap Ky$. Then C is isomorphic to a non-zero K-subgroup of K/P and also a non-zero K-subgroup of K/Q.

Therefore P is related to Q.

$Q \sim P \in \text{Ass}\, M$.

Therefore $\text{Ass}\, M = \{[P]\}$.

Definition 4.2—Let N be a submodule of M. N is said to be irreducible if for any two submodules N_1 and N_2, $N \underset{\neq}{\subset} N_1$, $N \underset{\neq}{\subset} N_2 \Rightarrow N \underset{\neq}{\subset} N_1 \cap N_2$.

And N is said to be 'strongly irreducible' if for any two K-subgroups N_1 and N_2, $N \underset{\neq}{\subset} N_1$ and $N \underset{\neq}{\subset} N_2 \Rightarrow N \underset{\neq}{\subset} N_1 \cap N_2$.

If M satisfies property (P), clearly a submodule N is irreducible if and only if it is strongly irreducible.

Hence if N is an irreducible submodule then M/N is uniform.

Theorem 4.3—If M is a K-module which satisfies A.C.C. on K-subgroups then every submodule of M is a finite intersection of irreducible submodules.

This can be proved by using the A.C.C. on K-subgroups.

Definition 4.4—A primary decomposition of a submodule N of M is a representation of N as a finite intersection of primary submodules.

Theorem 4.5—If M is a K-module satisfying A.C.C. on K-subgroups and property (p), then every submodule N of M has a primary decomposition.

Proof : By Theorem 4.3, a submodule N of M can be written as $N = N_1 \cap N_2 \cap \ldots \cap N_k$ where each N_i is irreducible submodule. By the property (p) each N_i is strongly irreducible.

Then for each i, M/N_i is uniform and satisfies A.C.C. on K-subgroups and it is coprimary by Theorem 4.1. Thus, N_i is primary.

5. Z-S-Coprimary Submodules

It can be shown that every primary decomposition has reduced primary decomposition and uniqueness by the familiar methods.

Lemma 5.1—Let K be a near-ring with 1 satisfying the conditions.

(a) Possesses A.C.C. on ideals.

(b) Every left K-subgroup is an ideal of K.

Then every critical left ideal of K is a prime ideals.

Proof : By condition (b), P is an ideal of K and by definition of critical left ideal, K/P is strongly uniform.

If P is not prime, there exists ideals A and B such that $P \subsetneq A$, $P \subsetneq B$ and $BA \subset P$. Since $P \neq A$, there exists $a \in A$ and $a \notin P$.

Define mapping $f : K/P \to K/P$ as follows :

$$f(x + P) = xa + P$$

clearly this is a module homomorphism and Ker $f \supset B/P$ and $f \neq 0$.

For if $f = 0$, then $xa \in P$ for every $x \in K$.

That is $Ka \subset P$ which is not the case since $a \in Ka$ and $a \notin P$.

This is a contradiction to the hypothesis that K/P is a rational extension of each of its non-zero K-subgroups.

Hence P is prime.

Definition 5.2—An ideal Q of K is said to be Z-S-coprimary if $Px \subset Q \Rightarrow P \subset r(Q)$ or $x \in Q$

where $r(Q)$ is the intersection of prime ideals.

Theorem 5.3—If K-satisfies conditions above, then an ideal Q which is Z-S-coprimary is coprimary in the sense of this article.

Proof : $P \in \text{Ass}(K/Q)$ and $P = \text{Ann}(x + Q)$.

That is $Px \subset Q$, $x \notin Q$.

Therefore $P \subset r(Q)$ but $r(Q) \subset P$ (since P is prime)

therefore $P = r(Q)$ which implies Ass $(K/Q) = \{[P]\}$.

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REFERENCES

1. M. N. Barua, Ph. D. Dissertation, Gauhati University, Assam, 1982.
2. G. Pilz, *Near-rings*, North Holland, New York (1983).
3. Hans H. Storrer, *Lecture Notes in Math.* **246** (1972), 617–62.

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SOME RESULTS ON ALMOST SEMI-INVARIANT SUBMANIFOLD OF AN *SP*-SASAKIAN MANIFOLD

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In the present paper I have obtained some of the properties of an almost semi-invariant submanifold of an *SP*-Sasakian manifold. The intergrability conditions of the distributions D, $D^\perp$, $\widetilde{D}$, $D \oplus \{\xi\}$, $D^\perp \oplus \{\xi\}$ and $\widetilde{D} \oplus \{\xi\}$ have also been discussed.

1. INTRODUCTION

Let $\widetilde{M}$ be an n dimensional C^∞-manifold. If there exists in $\widetilde{M}$ a tensor field F of type (1, 1) a vector field ξ and a 1-form η satisfying

$$F^2 X = X - \eta(X)\,\xi, \quad \eta(\xi) = 1. \qquad \text{...(1.1)}$$

Then $\widetilde{M}$ is called an almost paracontact manifold.

Let g be a Riemannian metric satisfying

$$\eta(X) = g(X, \xi) \qquad \text{...(1.2)}$$

$$\eta(FX) = 0, \quad F\xi = 0, \quad \text{rank}(F) = n - 1 \qquad \text{...(1.3)}$$

$$g(FX, FY) = g(X, Y) - \eta(X)\,\eta(Y). \qquad \text{...(1.4)}$$

Then the set (F, ξ, η, g) is called an almost paracontact Riemannian structure and the manifold is called an almost paracontact Riemannian manifold[2].

Moreover if we define

$$'F(X, Y) = g(FX, Y) \qquad \text{...(1.5)}$$

then

$$'F(X, Y) = {}'F(Y, X) \qquad \text{...(1.6a)}$$

$$'F(FX, FY) = {}'F(X, Y). \qquad \text{...(1.6b)}$$

Now, let us consider a manifold in which the 1-form η satisfies

$$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0 \qquad \text{...(1.7)}$$

$$(\nabla_X \nabla_Y \eta)(Z) = (-g(X, Z) + \eta(X)\,\eta(Z))\,\eta(Y) + (-g(X, Y) + \eta(X)\,\eta(Y))\,\eta(Z) \qquad \text{...(1.8)}$$

where ∇ denotes the covariant differentiation with respect to g. Furthermore, if we put

$$\eta(X) = g(X, \xi), \ \nabla_X \xi = FX \qquad \text{...(1.9)}$$

then it is easily verified that the manifold in consideration becomes an almost para-contact manifold. Such a manifold is called a p-Sasakian manifold[4].

If the 1-form η in $\widetilde{M}$ satisfies

$$(\nabla_X \eta)(Y) = -g(X, Y) + \eta(X)\eta(Y) \qquad \text{...(1.10)}$$

we can easily show by putting $\eta(X) = g(X, \xi)$ and $(\nabla_X \eta)(Y) = 'F(X, Y)$, that the manifold is p-Sasakian. Such a manifold is called an sp-Sasaksan manifold[4]. Thus in an sp-Sasakian manifold, we have

$$F(Y, Z) = -g(Y, Z) + \eta(Y)\eta(Z) \qquad \text{...(1.11)}$$

and

$$(\nabla_X 'F)(Y, Z) = (-g(X, Y) + \eta(X)\eta(Y))\eta(Z)$$
$$+ (-g(X, Z) + \eta(X)\eta(Z))\eta(Y). \qquad \text{...(1.12)}$$

Let M be an m-dimensional submanifold immersed in an sp-Sasakian manifold $\widetilde{M}$. Let TM and $TM^\perp$ be respectively the tangent and the normal bundle to M. Suppose the structure vector field ξ is tangent to M and denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M and $\{\xi\}^\perp$ the complementary orthogonal distribution to $\{\xi\}$ in TM. For each $X \in \Gamma(TM)$, put

$$FX = bX + cX \qquad \text{...(1.13)}$$

where $bX \in \Gamma(\{\xi\}^\perp)$ and $cX \in \Gamma(TM)^\perp$. Thus b is an endomorphism of the tangent bundle TM and c is a normal bundle valued 1-form on M.

Definition 1.1—The submanifold M of the sp-Sasakian manifold $\widetilde{M}$ is said to be an almost semi-invariant submanifold if its tangent bundle TM has the decomposition

$$TM = D \oplus D^\perp \oplus \widetilde{D} \oplus \{\xi\} \qquad \text{...(1.14)}$$

where

(1) D is invariant distribution on M, i.e.

$$F(D_x) = D_x$$

(2) $D^\perp$ is an anti-invariant distribution on M, i.e.

$$F(D_x^\perp) \subset T_x M^\perp \quad \text{for } x \in M.$$

(3) $\widetilde{D}$ is neither invariant nor an anti-invariant distribution on M, i.e. $b\,X_x \neq 0$ and $C\,X_x \neq 0$ for any $x \in M$ and $X_x \in D_x$.

(4) $\{\xi\}$ is the distribution spanned in M by the vector field ξ.

2. Basic Results

Let M be an almost semi-invariant submanifold of an sp-Sasakian manifold $\widetilde{M}$. We denote the Riemannian metrics of $\widetilde{M}$ and M both by g. Let P, Q and L be the projection morphisms of TM to the distributions D, $D^\perp$ and $\widetilde{D}$ respectively. Then for $X \in T(M)$, we have

$$X = PX + QX + LX + \eta(X)\,\xi. \qquad \ldots(2.1)$$

Now, we take $X \in \Gamma(\widetilde{D})$. Then $bX \neq 0$, $CX \neq 0$. Thus C defines a vector sub-bundle $C\widetilde{D} : x \to C\widetilde{D}_x$ of $TM^\perp$.

For any $N \in \Gamma(TM^\perp)$, we put

$$FN = tN + fN \qquad \ldots(2.2)$$

where tN and fN are respectively the tangential and the normal components of FN. Then we have,

$$g(F\,D^\perp,\ C\widetilde{D}) = 0. \qquad \ldots(2.3)$$

Next, we denote by ν the orthogonal complementary vector bundle to $FD^\perp \oplus C\widetilde{D}$ in $TM^\perp$. By (1.4), we have

$$g(FX, CY) = g(FX, FY) = g(X, Y) = 0 \;\forall\; X \in \Gamma(D^\perp)$$
$$Y \in \Gamma(\widetilde{D}). \qquad \ldots(2.4)$$

Thus

$$TM^\perp = FD^\perp \oplus C\widetilde{D} \oplus \nu. \qquad \ldots(2.5)$$

Lemma 2.1—The morphisms t and f satisfy

$$t(TM^\perp) = D^\perp \oplus \widetilde{D} \qquad \ldots(2.6)$$

$$t(FD^\perp) = D^\perp \qquad \ldots(2.7)$$

$$t(C\widetilde{D}) = \widetilde{D} \qquad \ldots(2.8)$$

$$f(C\widetilde{D}) = C\widetilde{D}. \qquad \ldots(2.9)$$

PROOF : Let $N \in \Gamma(TM^{\perp})$, then

$$g(tN, X) = g(FN, X) = g(N, FX) = 0 \ \forall \ X \in \Gamma(D)$$

and

$$g(tN, \xi) = g(FN, \xi) = g(N, F\xi) = 0.$$

Thus $tN \in \Gamma(D^{\perp} \oplus \widetilde{D})$ and we get (2.6). Next for each $X \in \Gamma(D^{\perp})$, we have

$$X = F^2 X = tFX + CFX = tFX$$

which implies (2.7). The proof of (2.8) and (2.9) is same as given in Bejancu and Papaghiuc[1].

Definition 2.1—An almost semi-invariant submanifold M in an *sp*-Sasakian manifold $\widetilde{M}$ is said to be a semi-invariant submanifold if we have $\widetilde{D} = \{0\}$.

Proposition 2.1—Let M be an almost semi-invariant submanifold of the *sp*-Sasakian manifold $\widetilde{M}$. Then the endomorphism $b : TM \to TM$ is a para-f structure on M, that is, $b^3 - b = 0$ if and only if M is a semi-invariant submanifold.

PROOF : From (1.13), we see that

$$(b^3 - b) X = 0 \quad \text{for any} \quad X \in \Gamma(D \oplus D^{\perp} \oplus \{\xi\}).$$

Since $b\, \widetilde{D}_x = \widetilde{D}_x$, we see that

$$(b^3 - b)(\widetilde{D}) = \{0\}$$

if and only if

$$(b^2 - I)(\widetilde{D}) = \{0\}.$$

Applying F to (1.13) and using (1.13), (2.2) and (1.1) we get

$$X = F^2 X = b^2 X + CbX + tCX + fCX \text{ for } X \in \Gamma(D).$$

Equating the tangent part, we get

$$(b^2 - I) = - tC.$$

Therefore,

$$tC(\widetilde{D}) = \{0\}$$

which with the help of (2.8) gives $\widetilde{D} = \{0\}$ Hence the submanifold is semi-invariant.

Proposition 2.2—Let M be an almost semi-invariant sub-manifold of the *sp*-Sasakian manifold $\widetilde{M}$. Then M is a semi-invariant submanifold if and only if

$$f^3 - f = 0. \qquad \text{...(2.10)}$$

PROOF : We see that if $N \in (FD^{\perp})$, we have $fN = 0$ and for $N \in \Gamma(\nu)$, $fN = FN$. By (2.9) we see that f is an automorphism on CD. Hence $(f^3 - f)(\widetilde{CD}) = \{0\}$ if and only if

$$(f^2 - I)(\widetilde{CD}) = \{0\}. \qquad \ldots(2.11)$$

By means of (1.13), (2.2) and (1.1), we get

$$CX = F^2(CX) = bfCX + tfCX + C + CX + f^2 CX \qquad \ldots(2.12)$$

for any $X \in (\widetilde{CD})$.

Thus (2.11) is equivalent to

$$Ct(\widetilde{CD}) = \{0\} \qquad \ldots(2.13)$$

which with the help of (2.8) gives (2.16).

Lemma 2.2—Let M be an almost semi-invariant submanifold of an sp-Sasakian manifold $\widetilde{M}$. Then we have

$$(b^2 + tC)X = X - \eta(X)\xi,\ (Cb + fC)X = 0 \qquad \ldots(2.14)$$

$$(bt + tf)N = 0,\ (f^2 - I + Ct)N = 0 \qquad \ldots(2.15)$$

$$(f^3 - f + Ctf)N = 0 \qquad \ldots(2.16)$$

$$(b^3 - b + tCb)X = 0 \qquad \ldots(2.17)$$

for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$.

PROOF : It follows directly from (1.1), (1.13) and (2.2).

Let $\widetilde{\nabla}$ (resp ∇) be the Riemannian connection on $\widetilde{M}$ (resp M) with respect to the Riemannian metric g. The linear connection induced by $\widetilde{\nabla}$ on the normal bundle $TM^{\perp}$ is denoted by $\nabla^{\perp}$, then the equations of Gauss and Weingarten are respectively given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \qquad \ldots(2.18)$$

and

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \qquad \ldots(2.19)$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, h is the second fundamental form of M and A_N is the fundamental tensor with respect to the normal section N and

$$g(h(X, Y), N) = g(A_N X, Y). \qquad \ldots(2.20)$$

Lemma 2.3—Let M be an almost semi-invariant submanifold of the sp-Sasakian manifold $\tilde{M}$. Then we have

$$P(u(X, Y)) = FP(\nabla_X Y) - \eta(Y) PX \quad \ldots(2.21)$$

$$Q(u(X, Y)) = Qt(h(X, Y)) - \eta(Y) Qx \quad \ldots(2.22)$$

$$L(u(X, Y)) = bL(\nabla_X Y) + Lt(h(X, Y)) - \eta(Y) LX \quad \ldots(2.23)$$

$$\eta(u(X, Y)) = -g(FX, FY) \quad \ldots(2.24)$$

$$h(X, FPY) + h(X, bLY) + \nabla^{\perp}_X FQY + \nabla^{\perp}_X CLY$$

$$= FQ \nabla_X Y + CL \nabla_X Y + f(h(X, Y)) \quad \ldots(2.25)$$

where

$$u(X, Y) = \nabla_X FPY + \nabla_X bLY - A_{FQY} X - A_{CLY} X \quad \ldots(2.26)$$

for all $X, Y \in \Gamma(TM)$.

The proof follows from Bejancu and Papaghiuc[1].

Lemma 2.4—Let M be an almost semi-invariant submanifold of an sp-Sasakian manifold M. Then we have

$$\nabla_X \xi = FX, \; h(X, \xi) = 0; \quad \text{for any } X \in \Gamma(D) \quad \ldots(2.27)$$

$$\nabla_Y \xi = 0, \; h(Y, \xi) = FY; \quad \text{for any } Y \in \Gamma(D^{\perp}) \quad \ldots(2.28)$$

$$\nabla_Z \xi = bZ, \; h(Z, \xi) = CZ, \quad \text{for any } Z \in \Gamma(\tilde{D}) \quad \ldots(2.29)$$

$$\nabla_\xi \xi = 0, \; h(\xi, \xi) = 0. \quad \ldots(2.30)$$

PROOF: We have

$$\tilde{\nabla}_X \xi = \nabla_X \xi + h(X, \xi) \quad \ldots(2.31)$$

which with the help of (1.9) and (2.1) gives

$$\nabla_X \xi + h(X, \xi) = FPX + FQX + bLX + CLX \quad \ldots(2.32)$$

for any $X \in \Gamma(TM)$.

Now, (2.27)–(2.30) follows from (2.32).

Lemma 2.5—Let M be an almost semi-invariant submanifold of the sp-Sasakian manifold $\tilde{M}$, then we have

$$A_{FX} Y + A_{FY} X = 0. \quad \ldots(2.33)$$

for all $X, Y \in \Gamma(D^{\perp})$.

PROOF : With the help of (1.4), (2.18), (2.20), we have

$$g(A_{FX}Y, Z) = g(h(Y, Z), FX) = g(\tilde{\nabla}_Z Y, FX) = g(F\tilde{\nabla}_Z Y, X)$$

$$= g(\tilde{\nabla}_Z FY, X) = -g(FY, \tilde{\nabla}_Z X) = -g(h(X, Z), FY)$$

$$= -g(A_{FY}X, Z)$$

for all $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(TM)$, which implies (2.33).

Lemma 2.6—Let M be an almost semi-invariant submanifold of the *sp*-Sasakian manifold $\tilde{M}$. Then we have

$$\nabla_\xi U \in \Gamma(D) \quad \text{for any } U \in \Gamma(D) \qquad \ldots(2.34)$$

$$\nabla_\xi V \in \Gamma(D^{\perp}) \quad \text{for any } V \in \Gamma(D^{\perp}) \qquad \ldots(2.35)$$

$$\nabla_\xi W \in \Gamma(\tilde{D}) \quad \text{for any } W \in \Gamma(\tilde{D}). \qquad \ldots(2.36)$$

The proof follows from Bejancu and Papaghiuc[1].

Corollary 2.1—Let M be an almost semi-invariant submanifold of the *sp*-Sasakian manifold $\tilde{M}$. Then we have

$$[X, \xi] \in \Gamma(D) \quad \text{for any } X \in \Gamma(D) \qquad \ldots(2.37)$$

$$[Y, \xi] \in \Gamma(D^{\perp}) \quad \text{for any } Y \in \Gamma(D^{\perp}) \qquad \ldots(2.38)$$

$$[Z, \xi] \in \Gamma(\tilde{D}) \quad \text{for any } Z \in \Gamma(\tilde{D}) \qquad \ldots(2.39)$$

The proof follows from Lemmas (2.4) and (2.6).

3. Integrability of Distributions

Theorem 3.1—Let M be an almost semi-invariant submanifold of an *sp*-Sasakian manifold $\tilde{M}$. Then the distribution D is integrable if and only if

$$h(X, FY) = h(Y, FX). \qquad \ldots(3.1)$$

PROOF : By using (2.27), we have

$$g([X, Y], \xi) = g(\nabla_X Y - \nabla_Y X, \xi)$$

$$= -g(\nabla_X \xi, Y) + \ldots + g(\nabla_Y \xi, X)$$

$$= -g(FX, Y) + g(FY, X)$$

$$= 0$$

for all $X, Y \in \Gamma(D)$.

Next, from (2.25), we have

$$h(X, FY) = FQ\, \nabla_X Y + CL\, \nabla_X Y + f(h(X, Y)) \qquad \ldots(3.2)$$

for any $X, Y \in \Gamma(D)$. Hence we have further

$$h(X, FY) - h(Y, FX) = FQ([X, Y]) + CL([X, Y]) \qquad \ldots(3.3)$$

which proves the theorem.

From Theorem 3.1 and (2.37) follows :

Corollary 3.1—The distribution $D \oplus \{\xi\}$ is integrable if and only if (3.1) is satisfied.

Theorem 3.2—The distribution $D^{\perp}$ is never integrable.

PROOF : For $X, Y \in \Gamma(D^{\perp})$, (2.26) gives

$$u(X, Y) = -A_{FY} X.$$

Applying F to (2.21) and using (1.1), we get

$$P(\nabla_X Y) = FP(A_{FY} X), \quad \text{for any} \quad X, Y \in \Gamma(D^{\perp})$$

which with the help of Lemma (2.5) gives

$$P([X, Y]) = FP(A_{FY} X - A_{FX} Y) = 2FP(A_{FY} X)$$

showing the non-integrability of $D^{\perp}$.

From Theorem 3.2 and (2.38) follows :

Corollary 3.2—The distribution $D^{\perp} \oplus \{\xi\}$ is never integrable.

Theorem 3.3—The distribution $\tilde{D}$ is integrable if and only if

$$A_{CY} X - A_{CX} Y - \nabla_X bY + \nabla_Y bX \in \Gamma(D^{\perp} \oplus \tilde{D} \oplus \{\xi\}) \qquad \ldots(3.4)$$

and

$$h(bX, Y) - h(X, bY) + \nabla^{\perp}_Y CX - \nabla^{\perp}_X CY \in \Gamma(C\tilde{D} + \nu) \qquad \ldots(3.5)$$

for any $X, Y \in \Gamma(\tilde{D})$.

PROOF : For any $X, Y \in \Gamma(\tilde{D})$, using (2.29), we get

$$\begin{aligned} g([X, Y], \xi) = g(\nabla_X Y - \nabla_Y X, \xi) &= g(X, \nabla_Y \xi) - g(Y, \nabla_X \xi) \\ &= g(X, bY) - g(Y, bX) \\ &= 0. \end{aligned} \qquad \ldots(3.6)$$

Now, for any $X, Y \in \Gamma(\tilde{D})$, (2.26) gives

$$u(X, Y) = \nabla_X bY - A_{CY} X. \qquad \ldots(3.7)$$

Applying F to (2.21) and using (1.1) and (3.7), we get

$$P(\nabla_X Y) = FP(A_{CY} X - \nabla_X bY) \qquad \ldots(3.8)$$

$$P([X, Y]) = FP(A_{CY} X - A_{CX} Y - \nabla_X bY + \nabla_Y bX) = 0 \qquad \ldots(3.9)$$

if and only if (3.4) is satisfied. Applying F to (2.25) and taking the components in $D^{\perp}$, we get

$$Q \nabla_X Y = Qt(h(X, bY) + \nabla^{\perp}_X CY - fh(X, Y))$$

which further yields

$$Q([X, Y]) = Qt(h(X, bY) - h(Y, bX) + \nabla^{\perp}_X CY - \nabla^{\perp}_Y CX).$$

Hence $\tilde{D}$ is integrable if and only if (3.5) is also satisfied.

From Theorem 3.3 and (2.39) follows :

Corollary 3.3—The distribution $\tilde{D} \oplus \{\xi\}$ is integrable if and only if (3.4) and (3.5) are satisfied simultaneously.

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References

1. A. Bejancu and N. Papaghiuc, *Bull. Math. de la Soc. Sci. Math. de la R.S. de Roumanie* **28** (76), nr 1 (1984).
2. J. Sato, *Tensor N. S.* **30** (1976), 219–24.
3. N Papaghiuc, *Bull. Math. de la Soc. Sci. Math. de la R S. de Roumanie* **28** (76) nr. 3 (1984).
4. T. Adati and T. Miyajawa, *TRU Math.* **13** (1977), 33–42.

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MAXIMAL ELEMENTS IN BANACH SPACES

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The existence of maximal elements is proved in closed, bounded and convex, but not necessarily compact, subsets of Banach spaces. The theorems proved in the paper are general enough to apply to the standard infinite-dimensional commodity spaces used in economic analysis.

1. INTRODUCTION

Suppose that K is a subset of a Hausdorff topological vector space E. Then each binary relation P on K gives rise to a multivalued map $T : K \to 2^K$ as follows: if $x \in K$, then $T(x) = \{y \in k : (x, y) \in P\}$. Conversely, if $T : K \to 2^K$ is a multivalued map, then a binary relation P on K is defined as follows: $(x, y) \in P$ if and only if $y \in T(x)$. A point x_0 of K is said to be a maximal element of the map $T : K \to 2^K$, with respect to the binary relation defined above, if $T(x_0) = \phi$.

Theorems on the existence of maximal elements have important applications in mathematical economics. For example, in recent work in general equilibrium theory without ordered preferences, the existence of an equilibrium in an abstract economy or qualitative game is often proved by constructing a map P, which may be construed as a 'preference map', on a subset K of a Hausdorff topological vector space and then by showing that there exists a point x_0 such that $P(x_0) = \phi$. See, for example, Aliprantis and Brown[1], Gale and Mas-Colell[10], Sonnenschein[18], Borglin and Keiding[5], Bergstrom[2], Schofield[16], Walker[21] and Toussaint[20]. In this work the assumption is made that the preference map is defined on a compact and convex subset of a finite-dimensional Cartesian space, or more generally, a Hausdorff topological vector space.

Some of these results have been generalized by Yannelis and Prabhakar[23] (Theorem 5.3) who have proved the existence of maximal elements in paracompact convex spaces and by Border[4] (Chapter 7) who has proved the existence of maximal elements in σ-compact convex spaces by using the idea of an escaping sequence.

For an extension of some of the results in the literature to the manifold setting the reader is referred to the interesting paper of Schofield[17]. Schofield's line of argument is somewhat removed from our main theme and will not be considered further here.

The object of this paper is to remove altogether the compactness assumptions on the domain and codomain of the 'preference map'. This is achieved by strengthening the assumptions on the 'preference map'. More precisely, we prove that a contracting 'preference map' (definitions follow) which satisfies the other usual conditions, has a maximal element in any closed bounded and convex but not necessarily compact subset of a Banach space. The theorms that we prove are general enough to apply to the standard infinite-dimensional commodity spaces used in economic analysis such as the sequence spaces l_p ($1 \leqslant p \leqslant \infty$), the Lebesgue spaces L_p ($1 \leqslant p \leqslant \infty$) and the space $M(K)$ of finite signed Baire measures on a compact metric space K, since these are all Banach spaces.

The economic motivation for the approach used in this paper will be made clearer by the following somewhat informal remarks. In R^n the existence of maximal elements is easily obtained as a consequence of natural economic assumptions (see Debru[7]). The argument goes as follows. Assume that income and all prices are strictly positive. Under these conditions the budget set is easily seen to be closed and bounded. Now in R^n, a closed and bounded set is compact. Hence if preferences are continuous (or even upper semicontinuous) in any of the equivalent topologies on R^n, there is a maximal element in the budget set and the demand correspondence is nonempty valued.

This argument fails in infinite-dimensional spaces because of their greater complexity. In a general topological vector space it is not always true that a closed and bounded set is compact. In fact, this property holds only for a very special class of spaces called semi-Montel spaces (see Wilansky[22], p. 90). Since every semi-Montel space is semi-reflecxive and a locally convex metric space is semi-reflexive if and only if it is reflexive, it follows that every non reflexive Banach space is not a semi-Montel space (see Wilansky[22], p. 153 and p. 265 and Duffie[8] for a discussion of semi-reflexive and semi-Montel spaces).

Let us consider the untoward implication of this fact for the space L_∞ which has been widely used in the economics literature. See, for example Bewley[3], Brown and Lewis[6] and Toussaint[20]. Since L_∞ is not reflexive (see Royden[15], p. 191) it is not a semi-Montel space. Hence the natural argument which shows that a closed and bounded set is compact does not extend to it and the existence of maximal elements is more difficult to prove (cf. Jones[12]).

The upshot of this discussion is that in infinite-dimensional spaces there is no 'natural' way in which one can prove the existence of maximal elements

In certain situations the following approach has been used to solve the problem Suppose that the commodity space E is the dual of some Banach space G. An example is the space L_∞ which is the dual of L_1. Under these conditions, the Alaoglu-Bourbaki theorem (see Wilansky[22], p. 130) asserts that a norm bounded set in E is relatively compact in the weak $*$ topology. This means that a norm bouded and

closed set in E is weak * compact. Hence, if preferences are continuous, every closed and bounded attainable set in E has a maximal elemnent. This technique of obtaining the compactness of closed and bounded sets by the use of a suitable topology and then proving the existence of maximal elements has been used by Florenzano[9].

Observe, however, that this approach only works in the case where the Banach commodity space has a predual. It does not work for general commodity spaces. Conditions for a Banach space to have a predual are given in Holmes[11] (pp. 211-14).

The object of this paper is to suggest and another way in which the existence of maximal elements can be obtained. This approach is more general since it does not require the Banach space to have a predual. In view of the difficulty of obtaining compact sets in infinite-dimensional spaces, it serves a heuristic purpose by suggesting that compactness may be needed to prove the existence of maximal elements in budget sets or attainable sets if preferences satisfy a contraction condition.

2. Preliminaries

We shall use the following notation. If K is a subset of a Banach space, then int K denotes the topological interior of K, co K denotes the convex haul of K and $\overline{\text{co}}\, K$ denotes the closed convex hull of K.

Let X be a Banach space and S a bounded subset of X. Then the Kuratowski 'measure of non-compactness' of S, $\alpha(S)$ is defined by

$$\alpha(S) = \inf\{\epsilon > 0 : S \text{ can be covered by a finite number of sets with diameter no larger than } \epsilon\}.$$

Observe that if S is compact, $\alpha(S) = 0$. The proofs of the following two theorms can be found in Lloyd[13], [Chapter 6].

Theorem (Kuratowski)—Suppose that (A_n) is a decreasing sequeuce of non-empty closed sets in a Banach space such that $\alpha_n = \alpha(A_n)$ tends to zero an n tends to infinity. Then $A = \bigcap_{n=1}^{\infty} A_n$ is non-empty and compact.

Theorem (Darbo)—If S is a bounded subset of a Banach space, then $\alpha(S) = \alpha(\overline{\text{co}}\, S)$.

Let Y_1, Y_2 be metric spaces. Then a multivalued map $T : Y_1 \to 2^{Y_2}$ is said to be a k-set contraction if for all bounded subsets S of Y_1, $T(S)$ is bounded and $\alpha(T(S)) \leq k\,\alpha(S)$. The map $T : Y_1 \to 2^{Y_2}$ is said to be a strict set contraction if it is a k-set contraction with $k < 1$.

Let K be a subset of a topological vector space and $T : K \to 2^K$ a multivalued map. We say that T is a preference map if it is generated from a binary relation P on

K so that $y \in T(x)$ if and only if $(x, y) \in P$. A point x in K is said to be a maximal element of the preference map T if $T(x) = \phi$.

3. Maximal Elements

We now prove the following theorem on the existence of maximal elements.

Theorem 1—Let E be a Banach space and D a non-empty closed, bounded and convex subset of E. Let the preference map $P : D \to 2^D$ satisfy the following conditions:

(i) for each $x \in D$, $P(x)$ is convex;

(ii) P is irreflexive, i.e. for each $x \in D$, $x \notin P(x)$;

(iii) for each $x \in D$ such that $P(x) \neq \phi$, there exists $y \in D$ such that $x \in \operatorname{int} P^{-1}(y)$;

(iv) the preference map P is a strict set contraction, i.e. $\alpha(P(S)) \leqslant t\,\alpha(S)$ for all bounded S, where $0 \leqslant t < 1$.

Then there exists a maximal element, i e. a point $x^* \in D$ such that $P(x^*) = \phi$.

Proof : Suppose that the theorem is false. Then $P(x) \neq \phi$ for every $x \in D$. As in Martin[14], [Chapter 4] the proof consists of two parts. In the first part, we show that condition (iv) implies the existence of a compact convex subset K of D such that P is a map from K into 2^K. In the second part of the proof, a fixed point theorem is applied to the map P.

Let $K_0 = D$ and for $n = 1, 2, \ldots$ $K_n = \overline{\text{co}}\,(P(K_{n-1}))$. We claim that for $n = 1, 2, \ldots$

$$K_n \subset K_{n-1} \text{ and } \alpha(K_n) \leqslant t^n \alpha(K_0) \quad (1)$$

Now $K_1 \subset K_0$ since D is closed and convex. Furthermore,

$$\alpha(K_1) = \alpha(\overline{\text{co}}\, P(K_0)) = \alpha(P(K_0)) \leqslant t\,\alpha(K_0)$$

where the second equality holds because of Darbo's theorem and the last inequality is a consequence of the fact that P is a strict set contraction. Hence, (1) hold for $n = 1$.

Assume that (1) holds for some $n \geqslant 1$. Then

$$K_{n+1} = \overline{\text{co}}\,(P(K_n)) \subset \overline{\text{co}}\,(P(K_{n-1})) = K_n$$

and

$$\alpha(K_{n+1}) = \alpha(\overline{\text{co}}\, P(K_n)) = \alpha(P(K_n) \leqslant t\,\alpha(K_n) \leqslant t^{n+1} \alpha(K_0)$$

so that (1) holds for all n and the proof of the claim is finished. Since K_n is a decreasing sequence of closed sets such that $\alpha(K_n) \to 0$ Kuratowski's theorems implies

that the set $K = \bigcap_{n=0}^{\infty} K_n$ is nonempty and compact. K is also convex as an intersection of convex sets.

Observe that $P(K_n) \subset P(K_{n-1}) \subset P(\overline{\text{co}}(K_{n-1})) = P(K_n)$ so that P maps K into 2^K, and the first part of the proof has been completed.

It only remains to check that the map $P : K \to 2^K$ satisfies all the conditions of the fixed-point theorm of Tarafdar[19]. Now $P(x) \neq \phi$ for every $x \in K \subset D$, by hypothesis, so that P is nonempty valued on K. Condition (i) implies that P is convex-valued on K. It only remains to prove that for each $x \in K$ there exists $y \in K$ such that $x \in \text{int}\, P^{-1}(y)$ in the relative topology of K. So suppose $x \in K \subset D$. Then condition (iii) implies that there exists $y \in D$ such that $x \in \text{int}\, P^{-1}(y)$. This means that $y \in P(x) \subset K$ so that $y \in K$. Hence, for each $x \in K$ there exists $y \in K$ such that $x \in \text{int}\, P^{-1}(y)$ where the interior is in D. Therefore, there exists an open neighbourhood N in D such that $x \in N \subset \text{int}\, P^{-1}(y)$. Since N is open in D, $N \cap K$ is open in K. Consequently, $x \in N \cap K \subset$ relative interior of $P^{-1}(y)$ in K. This proves that for each $x \in K$ there exists $y \in K$ such that $x \in \text{int}\, P^{-1}(y)$ in the relative topology of K.

Hence, from the fixed-point theorem of Tarafdar[19] we conclude that there exists a point $x_0 \in K$ such that $x \in P(x_0)$, contradicting condition (ii). The contradiction proves the theorem.

Corollary—Let D be a closed, bounded and convex subset of a Banach space. Suppose that the preference map $P : D \to 2^D$ satisfies the following conditions:

(i) for each $x \in D$, $P(x)$ is convex;

(ii) for each $x \in D$, $x \notin P(x)$;

(iii) for each $y \in D$, $P^{-1}(y)$ is open in D;

(iv) P is a strict set contraction.

Then there exists a maximal element.

PROOF : It is easily verified that if $P^{-1}(y)$ is open in D for $y \in D$, then condition (iii) of the the theorem is satisfied.

A map satisfying conditions (i), (ii) and (iii) of the corollary is a B-map. The following generalization of a B-map is due to Yannelis and Prabhakar[23].

Definition—A map $T : K \to 2^K$ where K is a subset of topological vector space E is said to be a BS map if the following conditions hold:

(i) for each $x \in K$, $x \notin \text{co}\, T(x)$;

(ii) $T^{-1}(y)$ is open in K for each $y \in K$.

Clearly every B-map is a BS map. The following theorem is due to Yannelis and Prabhakar[23], (Corollary 5.1).

Theorem 2—Let K be a compact convex subset of a Hausdorff topological vector space and $P : K \to 2^K$ a map that is locally BS-majorized (i.e. locally P is a sub-map of a BS map). Then P has a maximal element.

The assumption of the compactness of the domain can be weakened in Banach spaces if P is a strict set contraction. More precisely, we have the following theorem:

Theorem 3—Let D be a closed, bounded and convex subset of a Banach space E. Let the preference map $P : D \to 2^D$ satisfy the following conditions:

(i) P is a locally BS-majorised.

(ii) P is a strict set contraction.

Then there exists a maximal element.

PROOF : Suppose that the theorem is false. Then $P(x) \neq \phi$ for each $x \in D$. Proceeding as Theorem 1, we get a compact convex subset K of D such that $P : K \to 2^K$. Hence, $P(x) \neq \phi$ for every $x \in K$. Since P is locally BS-majorized, for each x, there exists a BS-map T_x ; $D \to 2^D$ such that $z \notin \text{co } T_x(z)$ for all $z \in D$ and an open neighbourhood U_x such that $z \in U_x$ implies $P(z) \subset T_x(z)$. Define $T'_x(z) = \text{co } T_x(z) \cap K$ and $U'_x = U_x \cap K$. Then $T_x : K \to 2^K$ is convex-valued and for every $z \in U'_x$, $P(z) \subset T'_x$. Clearly, $z \notin T'_x(z)$ for all $z \in K$. Hence, the map $P : K \to 2^K$ is locally BS-majorized and Theorem 2 implies that $P(x^*) = \phi$ for some x^*. The contradiction proves the theorem.

Remark : Since every locally B-majorized map is locally BS-majorized, it follows from Theorem 3 that every locally B-majorized preference map that is a strict set contraction has a maximal element in any closed, bounded and convex subset of a Banach space. This enables us to weaken the compactness condition of a Theorem 2.2 of Toussaint[20].

We turn now to the consideration of acyclic preference maps. First, we prove a generalization of a theorem of Bergstrom[2] and Walker[21].

Theorem 4—Let K be a compact subset of a topological space. Suppose the preference map $P : K \to 2^K$ satisfies the following conditions:

(i) P is acyclic; i.e. if

$$x_{i+1} \in P(x_i) \text{ for } i = 1, 2, \ldots, n \text{ then } x_1 \notin P(x_{n+1});$$

(ii) for each $x \in K$ such that $P(x) \neq \phi$, there exists $y \in K$ such $x \in \text{int } P^{-1}(y)$ in K.

Then there exists a maximal element.

Proof : Suppose that the theorem is false. Then $P(x) \neq \phi$ for each $x \in K$. This implies that for each $x \in K$ there exists $y \in K$ such that $y \in P(x)$. Hence, condition (ii) implies that for each $x \in K$ there exists $y \in K$ such that $x \in$ int $P^{-1}(y) = O_y$. The relatively open sets O_y cover K i.e. $K \cup O_y$. Since K is compact, there exists a finite subset $\{y_1, y_2, \ldots, y_n\}$ such that $K \bigcup_{i=1}^{n} O_{y_i}$.

It is easily verified that since P is acyclic, the set $\{y_1, y_2, \ldots, y_n\}$ has a maximal element $y' \in \{y_1, y_2, \ldots, y_n\}$, i.e. $y_i \notin P(y')$ for $i = 1, 2, \ldots, n$. Hence $y' \notin P^{-1}(y_i)$ (a fortiori, $y' \notin O_{y_i}$) for $i = 1, 2, \ldots, n$. This is a contradiction since $K = \bigcup_{i=1}^{n} O_{y_i}$, $y' \in K$ and $y' \notin O_{y_i}$ for $i = 1, 2, \ldots, n$. The contradiction proves the theorem.

Remark : If we assume that $P^{-1}(y)$ is open for each $y \in K$, we get the theorem of Bergstrom and Walker which is special case of Theorem 4 above.

For contracting preference maps the assumption of the compactness of the domain can be weakened. More precisely we have the following theorem.

Theorem 5—Let D be a closed bounded and convex subset of a Banach space E. Suppose that the preference map $P : D \to 2^D$ satisfies the following conditions:

(i) P is acyclic;

(ii) for each $x \in D$ such $P(x) \neq \phi$, there exists $y \in D$ such that $x \in$ int $P^{-1}(y)$ in D;

(iii) P is a strict set contraction.

Then there exists a maximal element.

Proof : Suppose that the theorem is false. Then $P(x) \neq \phi$ for every $x \in D$. Proceeding as in the first part of Theorem 1 we get a compact convex set K such that $P : K \to 2^K$. Arguing as in the second part of Theorem 1 we can prove that for each $x \in K$ there exists $y \in K$ such that $x \in$ int $P^{-1}(y)$ where the interior is in the relative topology of K. Hence, condition (ii) of Theorem 4 is satisfied. The restriction of P to K is clearly acyclic. This implies that condition (i) of Theorem 4 is satisfied. Consequently, Theorem 4 implies that there exists a point $x_0 \in K$ such that $P(x_0) = \phi$. The contradiction proves the theorem.

References

1. C. D. Aliprantis and B. Brown, *J. Math. Econ.* **11** (1983), 189–207.
2. T. Bergstorm, *J. Econ. Theory* **10** (1975), 403–404.
3. T. Bewley, *J. Econ. Theory* **4** (1972), 514–40.
4. K. Border, *Fixed Point Theorems with Applications to Economics and Game Theory*. Cambridge University Press, London, 1985.

5. A. Borglin and H. Keiding, *J. Math. Econ.* **3** (1976), 313-16.
6. D. Brown and L. Lewis, *Econometrica* **49** (1981), 359-68.
7. G. Debreu, *Theory of Value*. John Wiley and Sons, New York, 1959.
8. D. Duffie, *J. Math. Econ.* **14** (1986), 1-23.
9. M. Florenzano, *J. Math. Econ.* **12** (1983), 207-19.
10. D. Gale and A. Mas-Colell, *J. Math. Econ.* **2** (1975), 9-15.
11. R. Holmes, *Geometric Functional Analysis and Its Applications*. Springer-Verlag, New York, 1975.
12. L. Jones, *Models of Economic Dynamics* (ed.: H. Sonnenschein) Springer-Verlag, New York, 1986.
13. N. Lloyd, *Degree Theory*. Cambridge University Press, London, 1978.
14. R. Martin, *Non-linear Operators and Differential Equations in Banach Spaces*. John Wiley and Sons, London, 1976.
15. H. Royden, *Real Analysis* Macmillan, London, 1968.
16. N. Schofield, *Mathematical Methods in Economics*. Croom Helm, London, 1984.
17. N. Schofield, *Math. Oper. Res.* **9** (1984), 545-57.
18. H. Sonnenschein, in *Preferences, Utility and Demand* (eds: J. Chipman *et al.*) Harcourt Brace and Jovanovich, New York, 1971.
19. E. Tarafdar, *Proc. Amer. Math. Soc.* **67** (1977), 95-98.
20. S. Toussaint, *J. Econ. Theory* **33** (1984), 98-115.
21. M. Walker, *J. Econ. Theory* **16** (1977), 470-74.
22. A. Wilansky, *Modern Methods in Topological Vector Spaces*. McGraw-Hill Book Co., Inc., New York, 1978.
23. N. Yannelis and N. Prabhakar, *J. Math. Econ.* **12** (1983), 233-45.
24. N. Yannelis and W. Zame, *J. Math. Econ.* **15** (1986), 85-110.

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ON THE ENDL-TYPE GENERALIZATION OF CERTAIN SUMMABILITY METHODS

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It is shown that if A is a regular Euler, Taylor or Meyer-König matrix and $r > 0$, then A and its Endl-type generalization A^r are absolutely equivalent for all sequences $s_n = o(n^{1/2})$.

Endl[4] and Jakimovski[6] introduced the "Endl-type" generalization of Hausdorff and quasi-Hausdorff matrices and, more recently, Kuttner and Parameswaran[7] introduced the generalization of the same type of the Meyer-König — Ramanujan matrix. The Hausdorff, quasi-Hausdorff and Meyer-König—Ramanujan matrices, when they are conservative, may be considered as "built" around the Euler, Taylor and Meyer-König methods $A = E_\alpha$, T_α and S_α respectively and reflect some of the properties of these methods. A similar statement holds good when we consider the Endl-type generalizations of all these methods. It is the object of this note to study some properties of the Endl-type generalization A^r, where $A = E_\alpha$, T_α or S_α. It is shown that A and A^r $(r > 0)$ are absolutely equivalent for all sequences $s_n = o(n^{1/2})$ (see below for the definitions).

Definitions—(1) Let $0 < \alpha < 1$ and $r \geqslant 0$. The generalized Euler matrix $E^r_\alpha = \left(a^r_{nk}\right)$ is defined by

$$a^r_{nk} = \begin{cases} \binom{n+r}{n-k} \alpha^{k+r} (1-\alpha)^{n-k} & (k \leqslant n) \\ 0 & (k > n) \end{cases} \quad \ldots(1)$$

$n, k = 0, 1, \ldots, \ldots$ when $r = 0$ this gives the Euler matrix E_α. We write a_{nk} for a^0_{nk}

Let $0 < \alpha < 1$ and $r \geqslant 0$. The generalized Taylor matrix $T^r_\alpha = \left(b^r_{mn}\right)$ is defined by

$$b^r_{mn} = \begin{cases} \binom{n+r}{n-m} \alpha^{n-m} (1-\alpha)^{m+r+1} & (n \geqslant m \geqslant 0) \\ 0 & (n < m). \end{cases} \quad \ldots(2)$$

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When $r = 0$, this gives the Taylor matrix T_α. We write b_{mn} for b^0_{mn}.

(3) Let $0 < \alpha < 1$ and $r \geqslant 0$. The generalized Meyer-König matrix $S^r_\alpha = \left(c^r_{mn} \right)$ is defined by

$$c^r_{mn} = c_n(m + r) = \binom{m+r+n}{n} (1-\alpha)^{m+r+1} \alpha^n \quad (m, n \geqslant 0). \qquad \ldots(3)$$

When $r = 0$, this gives the Meyer-König matrix S_α. We write c_{mn} for c^0_{mn}. {For the definitions of T_α, S_α given above and further properties of these matrices, see Meyer-König[10]}.

(4) Following Cooke[2,3] we shall say that two given matrices A and B are absolutely equivalent for sequences s of a class X if $As - Bs \in (c_0)$ whenever $s \in X$. {For the matrices and sequences we shall discuss in this note, this notion coincides with "volläquivalenz" as defined in Zeller and Beekmann[15]}. Absolute equivalence with respect to fixed class X of sequences is obviously an equivalence relation and is in particular transitive.

Theorem 1—Let $0 < \alpha < 1$ and $r > 0$. Then E_α and E^r_α are absolutely equivalent for all sequences $s_n = o(n^{1/2})$.

PROOF : Let $s_n = o(n^{1/2})$, $E_\alpha s = t = \{t_n\}$ and $E^r_\alpha s = u = \{u_n\}$.

Now

$$| t_n - u_n | = | \Sigma a_{nk} s_k - \Sigma a^r_{nk} s_k | \leqslant \Sigma | a_{nk} - a^r_{nk} | | s_k | . \qquad \ldots(4)$$

(Throughout the proof of Theorem 1, the symbol Σ stands for $\sum_{k=0}^{n}$.) It is enough to prove that the sum in (4) tends to 0 as $n \to \infty$.

Case I : $r = 1$—Since $a^1_{nk}\ a_{n+1,k+1}$, the sum in (4) is

$$\Sigma | a_{nk} - a_{n+1,k+1} | | s_k | \leqslant \Sigma | a_{nk} - a_{n,k+1} | | s_k |$$
$$+ \Sigma | a_{n,k+1} - a_{n+1,k+1} | | s_k |$$

$= A_n + B_n$ (say). But $s_k = o(k^{1/2})$ and hence (i) $A_n \to 0$ (see Lorentz[9], Theorem 2 and §5.2 or Parameswaran[11], Theorem 3) and (ii) $B_n \to 0$ (Parameswaran[12], Lemma 5). Hence the theorem is true when $r = 1$.

Case II: *r is a positive integer* > 1—Now the sum in (4) is

$$\Sigma \mid a_{nk} - a_{n+r,k+r} \| s_k \mid \leqslant \sum_{t=0}^{r-1} \Sigma \mid a_{n+t,k+t} - a_{n+t+1,k+t+1} \| s_k \mid$$

$\to 0$ an $n \to \infty$ by repeated applications of Case I to $\{s_n\}$ and its translates.

Case III: $0 < r < 1$—For arbitrary fixed integers $n \geqslant k \geqslant 0$, let

$$a_{nk}(x) = \binom{n+x}{n-k} \alpha^{k+x} (1-\alpha)^{n-k} \qquad \text{.. (5)}$$

$$= \frac{(n+x)(n-1+x)\ldots(k+1+x)\,\alpha^{k+x}(1-\alpha)^{n-k}}{(n-)!}.$$

Then

$$\log a_{nk}(x) = \sum_{\nu=k+1}^{n} \log(x+\nu) - \log(n-k)!$$

$$+ (k+x)\log\alpha + (n-k)\log(1-\alpha)$$

and

$$\frac{a'_{nk}(x)}{a_{nk}(x)} = \sum_{\nu=k+1}^{n} \frac{1}{x+\nu} + \log\alpha. \qquad \text{.. (6)}$$

Since $\log \alpha < 0$ ang $\sum_{\nu=k+1}^{n} \frac{1}{x+\nu}$ is a positive decreasing function of x which tends to $+\infty$ as $x \to -(k+1)$ and to 0 as $x \to +\infty$, there exists $x_0 = x_0(n, k)$ such that the right-hand side of (6) is positive for $-(k+1) < x < x_0$ and negative for $x > x_0$. Thus

$$\left.\begin{array}{l}\text{the function } a_{nk}(x) \text{ is strictly increasing for} \\ -(k+1) < x < x_0 \text{ and strictly decreasing for } x > x_0.\end{array}\right\} (*)$$

Now, for any $t > -k$, $\dfrac{a_{nk}(t)}{a_{nk}(t-1)} = \dfrac{\alpha(n+t)}{k+t}$

and this equals 1 exactly when $t = (\alpha n - k)/(1-\alpha)$. It follows from (*) that $t - 1 < x_0 = x_0(n, k) < t$; that is,

$$\frac{\alpha n - k}{1-\alpha} - 1 < x_0(n, k) < \frac{\alpha n - k}{1-\alpha}. \qquad \text{...(7)}$$

Hence

$$a_{nk} = a_{nk}(0) < a_{nk}(x) < a_{nk}(1) = a_{n+1,k+1} \text{ for all } x \in (0, 1) \text{ if } 1 < x_0$$

and hence if $1 \leqslant (1-\alpha)^{-1}(\alpha n - k) - 1$. Thus

$$0 < a_{nk}(x) - a_{nk} < a_{n+1,k+1} - a_{nk}$$

if

$$\frac{\alpha n - k}{1 - \alpha} \geqslant 2. \qquad \ldots(8)$$

Similarly $a_{nk} = a_{nk}(0) > a_{nk}(x) > a_{nk}(1) = a_{n+1,k+1}$ for all $x \in (0, 1)$ if $0 > x_0$, and hence if

$$\frac{\alpha n - k}{1 - \alpha} \leqslant 0. \qquad \ldots(9)$$

Thus if n, k satisfy (8) or (9), then

$$\left.\begin{array}{l} | a_{nk}(x) - a_{nk} | < | a_{n+1,k+1} - a_{nk} | \\ \text{for all } x \in (0, 1). \end{array}\right\}. \qquad \ldots(10)$$

The inequality (10) may fail to hold only for those n, k such that neither (8) nor (9) holds; thus (10) holds except possibly when

$$0 < \frac{\alpha n - k}{1 - \alpha} < 2. \qquad \ldots(11)$$

We note that for each n, there are at most two the values of k for which (11) holds.

We write the sum in (4) as

$$\Sigma \, | a_{nk}(r) - a_{nk} || s_k | \leqslant \Sigma_1 + \Sigma_2$$

where Σ_1 denotes the sum taken over those k which satisfy (10) and Σ_2 denotes the sum over those k which satisfy (11). Then

$\Sigma_1 \leqslant \Sigma \, | a_{nk} - a_{n+1,k+1} || s_k | \to 0$ (as $n \to \infty$) as seen in Case I above; and Σ_2 is the sum of at most two terms which are $o(1)$ since $s_n = o(n^{1/2})$ and $\max\limits_k a_{nk}(x) = O(n^{-1/2})$ for each fixed $x > 0$ (see e. g. Hardy[5], Theorem 138 (2) or Lorentz[9], p. 313). Hence $\Sigma_2 \to 0$ as $n \to \infty$, and the theorem is proved for the case $0 < r < 1$ also.

Case IV—$r = [r] + q$, where $1 \leqslant [r]$ = the largest integer less than equal to r and $0 < q < 1$. Then by (1)

$$\Sigma \, | a^{r}_{nk} - a^{[r]}_{nk} || s_k |$$

$$= \Sigma \, | a^{q}_{n+[r],k+[r]} - a_{n+[r],k+[r]} || s_k |$$

$\to 0$ by Case III applied to the sequence $\{s_{k-[r]}\}$ (where, as usual, s_j is defined to be 0 when j is a negative integer). Thus E^{r}_{α} and $E^{[r]}_{\alpha}$ are absolutely equivalent for s and since E_α and $E^{[r]}_{\alpha}$ are absolutely equivalent for s by Case II, we see that the theorem is true for the Case IV also. This completes the proof of the theorem.

Definition 5—We say that a matrix method A is absolutely regular for a sequence

s if $As - As^* \in (c_0)$ where $s_n^* = s_{n-1}$ $(n \geqslant 0)$, or equivalently if

$$\sum_{k=0}^{\infty} a_{nk}\, s_k - \sum_{k=0}^{\infty} a_{n,k+1}\, s_k \to 0 \text{ as } n \to \infty \text{ (Cooke}^{2\cdot 3}\text{)}.$$

Theorem 2—The method E_α^r, where $0 < \alpha < 1$ and $r \geqslant 0$, is absolutely regular for all $s_n = o\,(n^{1/2})$

Proof : It is enough to prove that

$$\rho_n \equiv \Sigma \mid a_{nk}^r - a_{n,k+1}^r \mid \mid s_k \mid \to 0 \text{ if } s_n = o\,(n^{1/2}). \qquad \ldots(12)$$

This is true when $r = 0$ (by Theorems 2 and 9 (ii) of Lorentz[9]. Since

$$0 \leqslant \rho_n \leqslant \Sigma \;\{ \mid a_{nk}^r - a_{nk} \mid + \mid a_{nk} - a_{n,k+1} \mid + \mid a_{n,k+1}$$

$$- a_{n,k+1}^r \mid \} \mid s_k \mid$$

the relation (12) follows from Theorem 1.

Lemma 1—For $0 < t < 1$ and $r \geqslant 0$, let $\{E_n^r \;(t; s)\}$ denote the E_t^r -transform of $s = \{s_n\}$. Then

$$E_n^r\;(t; s) - E_{n-1}^r\,(t; s) = t\,\Sigma_{n-1}^r\,(t; \bar{s} - s)$$

where $\bar{s} = \{s_{n+1}\}$.

The proof for the case $r = 0$ has been given in Parameswarn[12]; the proof for the case $r > 0$ is similar.

Theorem 3— Let $A = (H^r, g)$ be the generalized Hausdorff matrix defined by

$$a_{nk} = \binom{n+r}{n-k} \int_0^1 t^{+r}\,(1 - t)^{n-k}\, dg\,(t)$$

where $g \in BV\,[0, 1]$ and $r \geqslant 0$, and let $s = \{s_n\} = \{\sum_0^n a_1\}$ satisfy the conditions $s_n = o\,(n^{1/2})$ and $a_n = O\,(1)$. Then the set of limit points of $u = As$ will be connected if $g\,(1) = g\,(1) = g\,(1 - 0)$.

Remark : Theorem 3 for the case $r = 0$ was proved by the author in Parameswaran[13]; Liu and Rhoades[8] proved the theorem for bounded $\{s_n\}$ and $r > 0$; (see also

Kuttner and Parameswaran[7]). Proofs in the more special case $s_n = O(1)$ and $r = 0$ have been given by various authors; Liu and Rhoades[8] or Parameswaran[13] for detailed references.

PROOF OF THEOREM 3 : Since $u_n = \int_0^1 E_n^r\,(t: s)\,dg\,(t)$ we have for $n \geqslant 1$,

$$u_n - u_{n-1} = \int_0^1 t\,E_{n-1}^r\,(t, a)\,dg\,(t) = \int_0^{1-0} + \int_{1-0}^1 \qquad ...(13)$$
$$= o\,(1) + [g\,(1) - g\,(1-0)]\,a_n$$

using Lemma 1 and the facts thnt $a_n = O\,(1)$ implies that $t\,E_{n-1}^r\,(t,\ a)$ is uniformly bounded in [0, 1] and tends to 0 as $n \to \infty$ (since $s_n = o\,(n^{1/2})$ implies that $\{a_n\}$ is E_t-summable to 0 (Hardy[5], p. 213) and hence is E_t^r -summable by Theorem 1). It follows from (13) that $u_n - u_{n-1} = o\,(1)$ if $g\,(1) = g\,(1-0)$ and hence the set of limit points of the bounded sequence $u = As$ is connected, by a theorem due to Barone[1].

Taeorim 4—Let $0 < \alpha < 1$ and $r > 0$. Then S_α and S^r_σ are absolutely equivalent for all sequences $s_n = o\,(n^{1/2})$.

PROOF : The proof is similar to that of Theorem 1 and we omit some of the details. Let $s_n = o\,(n^{1/2})$, $t = S_\alpha\ s$, $u = S_\alpha^r\ s$, $\bar{s} = \{s_{n+1}\}$, $\bar{t} = S_\alpha\,\bar{s}$ and $a_n = s_n - s_{n-1}$.

Case I : r is a positivc integer—Then $u_m = t_{m+r}$ by (3) and hence $u_m - t_m = \sum_{n=m+1}^{m+r} (t_n - t_{n-1})$. Now, as noted by (Meyer-König[10], p. 275),

$$t_n - t_{n-1} = \alpha\,(1 - \alpha^n \sum_{\nu=0}^{\infty} \left(\frac{n+\nu}{\nu}\right) \alpha^\nu\ a_{\nu+1}$$

$$= \left(\frac{\alpha}{1-\alpha}\right)(1-\alpha)^{n+1} \sum_{\nu=0}^{\infty} \binom{n+\nu}{\nu} \alpha^\nu\,(s_{\nu+1} - s_\nu)$$

$$= \left(\frac{\alpha}{1-\alpha}\right)\ (\bar{t}_n - t_n).$$ But $t_n - t_n). = o\,(1)$ by the the absolute regularity of sequences $s_n = o\,(^{1/2})$ (Parmeswaran[11], Theorem 3). Since r is a fixed positive integer, it follows that $u_m - t_m = o\,(1)$.

Case II : $0 < r < 1$—It is enough to prove that

$$\sum_{n=0}^{\infty} | c_n(m+r) - c_n(m) || s_n | = o(1). \qquad \ldots(14)$$

We follow Kuttner's argument used by Sitaraman[14], namely that

$$| c_n(m+r) - c_n(m) | < | c_n(m) - c_n(m+1) | \qquad \ldots(15)$$

except possibly for those values of n for which

$$m\alpha (1-\alpha)^{-1} < n < (m+2)\alpha(1-\alpha)^{-1}. \qquad \ldots(16)$$

We write the sum in (14) as $\Sigma_1, + \Sigma_2$, where Σ_1 is the contribution to the sum by those n which satisfy (16) and Σ_2 is the contribution by those n which do not satisfy (16). Now the number $g(m)$ of integers n satisfying (16) is a bounded function of m and hence it follows from the well known fact that $\max_n c_n(x) = O(x^{-1/2})$ as $x \to \infty$, that

$$\Sigma_1 \to 0 \text{ as } m \to \infty. \qquad \ldots(17)$$

Now $| \Sigma_2 | \leqslant \sum_{n=0}^{\infty} | c_n(m) - c_n(m+1) || s_n | = \sum_{n=0}^{\infty} | c_{mn} - c_{m+1,n} || s_n | = \rho_m$ (say), and

$$\sum_{n=0}^{\infty} c_{mn} - c_{m-1,n}) s_n = o(1) \text{ whenever } s_n = o(n^{1/2}) \qquad \ldots(18)$$

(as we observed in the proof of Case I). But (18) holds if and only $\rho_m = o(1)$ Cooke[2], Theorem 6 or Cooke[3], Theorem 5.51). It follows that $\Sigma_2 = o(1)$. The conclusion (14) now follows from (15) and (17).

Case III : $r > 1$—The desired result follows from Cases I and II since the mth term of $S_\alpha^r s$ is the $(m + [r])$th term of $S_\alpha^{r-[r]}$ s, where $[r]$ = the integral part of r.

Theorem 5—Let $0 < \alpha < 1$ and $r > 0$. Then T_α and T_α^r are absolutely equivalent for sequences $s_n = o(n^{1/2})$.

PROOF : We note that the relations (2) and (3) imply that

$$b^r_{mn} = 0 \text{ for } n < m \text{ and } b^r_{m,m+n} = c^r_{mn} \text{ for all } m, n.$$

Hence

$$\sum_{n=0}^{\infty} (b^r_{mn} - b_{mn}) s_n = \sum_{n=0}^{\infty} (b^r_{m,m+n} - b_{m,m+n}) s_{m+n}$$

$$= \sum_{n=0}^{\infty} (c^r_{mn} - c_{mn}) s_{m+n}. \qquad \ldots(21)$$

We may now prove the theorem either by using arguments similar to those used in the proof of Theorem 4, or, as suggested by the referee, we can derive the result from Theorem 4 as follows.

We note that if (α_{mn}), (β_{mn}) are regular matrices and if $\{\lambda_n\}$ is a given sequence of positive numbers then, in order that (α_{mn}). (β_{mn}) should be absolutely equivalent for sequences satisfying $s_n = o(\lambda_n)$, it is necessary and sufficient that

$$\sum_{n=0}^{\infty} | \alpha_{mn} - \beta_{mn} | \lambda_n = O(1). \qquad \ldots(22)$$

For we require that the transformation

$$t_m = \sum_{n=0}^{\infty} (\alpha_{mn} - \beta_{mn}) \lambda_n (s_n/\lambda_n)$$ regarded as a transformation from $\{u_n\} = \{s_n/\lambda_n\}$ to $\{t_m\}$, should transform null sequences into null sequences. The "standard" necessary and sufficient conditions for this are that (22) should hold and that, for fixed n,

$$\alpha_{mn} - \beta_{mn} \to 0 \text{ as } m \to \infty. \qquad \ldots(23)$$

But if (α_{mn}), (β_{mn}) are regular, then (23) is necessarily satisfied. Thus, Theorem 4 is equivalent to the assertion that

$$\sum_{n=0}^{\infty} | c^r_{mn} - c_{mn} | n^{1/2} = O(1). \qquad \ldots(24)$$

and, by equation (21), Theorem 5 is equivalent to the assertion that

$$\sum_{n=0}^{\infty} | c^r_{mn} - c_{mn} | (m+n)^{1/2} = O(1). \qquad \ldots(25)$$

Since Theorem 4 has been proved, we know that (24) holds. We have to prove that(25) also holds.

Now choose a constant A with $0 < A < \alpha/(1-\alpha)$. Then

$$\sum_{n<Am} c_{mn} (m+n)^{1/2} = O(e^{-\gamma m}) \qquad \ldots(26)$$

and

$$\sum_{n<Am} c^r_{mn} (m+n)^{1/2} = O(e^{-\gamma m}) \qquad \ldots(27)$$

where γ is a positive constant. The first of these results is given by Theorem 139 (2) of Hardy[5] the second may be proved in a similar way. But $(m+n)^{1/2} = O(n^{1/2})$ uniformly in $n \geqslant Am$ and hence, by (24),

$$\sum_{n \geqslant Am} | c^{r}_{mn} - c_{mn} | (m+n)^{1/2} = O(1). \qquad \ldots(28)$$

The required result (25) now follows from (26), (27) and (28).

References

1. H. G. Barone, *Duke Math. J.* **5** (1939), 740–52.
2. R. G. Cooke, *Proc. London Math. Soc.* (2) **41** (1936), 113–25.
3. R. G. Cooke, *Infinite Matrices and Sequence Spaces*. Macmillan and Co. London, 1950.
4. K. Endl, *Math. Annalen* **139** (1960), 403–32.
5. G. H. Hardy, *Divergent Series*, Oxford Univ. Press Oxford, 1949.
6. A. Jakimovski, *Technical Report No. 8, Jerusalem*, 1959.
7. B. Kuttner and M. R. Parameswaran, *Houston. J. Math.* 4 (1978), 569–76.
8. M. Liu and B. E. Rhoades, *Houston J. Math.* 2 (1976), 239–50.
9. G. G. Lorentz, *Canadian J. Math.* **1** (1949), 305–19.
10. W. Meyer-König, *Math. Z.* **52** (1949), 257–304.
11. M. R. Parameswaran, *Proc. Nat. Inst. Sci. India* **A25** (1959), 171–75.
12. M. R. Parameswaran, *J. Indian Math. Soc.* (**NS**) **23** (1959), 46–64.
13. M. R. Parameswaran, *Canadian Math. Bull.* **23** (1980), 137–40
14. Y. Sitaraman, *Math. Z.* **106** (1968), 153–57.
15. K. Zeller and W. Beekmann, *Theorie der Limitierungsverfahren*. Springer-Verlag Berlin 1970.

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MATRIX TRANSFORMATIONS IN SOME SEQUENCE SPACES

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The purpose of this paper is to characterise the matrices in the classes $(c(p), C_s)$, $(l_\infty(p), C_s)$ and $(l(p), C_s)$.

Let X and Y be any two nonempty subsets of the space of all sequences of complex numbers and let $A = (a_{nk})$, $(n, k = 1, 2, \ldots)$ be an infinite matrix of complex numbers. We write $Ax = (A_n(x))$ if $A_n(x) = \sum_k a_{nk} x_k$ converges for each n. (Throughout summation without limits runs from 1 to ∞). If $x = (x_k) \in X$ implies that $Ax = (A_n(x)) \in Y$, we say that A defines a matrix transformation from X into Y and we denote it by $A : X \to Y$. By (X, Y) we mean the class of matrices A such that $A : X \to Y$. If in X and Y there is some notion of limit or sum, then we write (X, Y, P) to denote the subset of (X, Y) which preserves the limit or sum.

If $p_k > 0$ and $\sup_k p_k < \infty$, we define (see Maddox[3])

$$c(p) = \{x : |x_k - l|^{p_k} \to 0 \text{ for some } l\}$$

$$l_\infty(p) = \{x : \sup_k |x_k|^{p_k} < \infty\}$$

$$l(p) = \{x : \sum_k |x_k|^{p_k} < \infty\}.$$

We define (see Stieglitz and Tietz[5])

$$C_S = \{x : \{\sum_{i=1}^{n} x_i\} \text{ is convergent}\}.$$

The purpose of this paper is to characterise the matrices in the classes $(c(p), C_S)$, $(l_\infty(p), C_S)$ and $(l(p), C_S)$.

The following notations are used throughout. For all integers $n \geqslant 1$ we write

$$t_n(Ax) = \sum_{i=1}^{n} A_i(x) = \sum b_{nk} x_k$$

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where

$$b_{nk} = \sum_{i=1}^{n} a_{ik}.$$

For all integers n, $t \geqslant 1$ and $1 \leqslant r \leqslant \infty$, we write

$$C(n, B, t, r) = \sum_{k=t}^{r} | b_{nk} |^{q_k} B^{-q_k}$$

where B is an integer and $p_k^{-1} + q_k^{-1} = 1$. We put

$$C(n, B, 1, \infty) = C(n, B) \text{ and } C(B) = \sup_n C(n, B).$$

We have

Theorem 1—$A \in (c(p), C_S)$ if and only if

(i) $D = \sup_n \sum_n | b_{nk} | B^{-1/p_k} < \infty$ for some integer $B > 1$

(ii) $\exists\, \alpha_k \in C$ such that

$$\lim_{n\to\infty} b_{nk} = \alpha_k \,(\forall\, k)$$

(iii) $\exists\, \alpha \in C$ such that

$$\lim_{n\to\infty} \sum b_{nk} = \alpha.$$

Proof : *Necessity*—Let $A \in (c(p), C_S)$. Put $t_n(Ax) = \sigma_n(x)$.

Since $(c(p), C_S) \subset (c_0(p), C_S)$, $\{\sigma_n(x)\}$ is a sequence of continuous linear functionals on $c_0(p)$ (see Maddox[2]) such that $\lim \sigma_n(x)$ exists. Therefore by uniform boundedness principle for $0 < \delta < 1$, there exist $S_\delta[0] \subset c_0(p)$ and a constant K such that $\sigma_n(x) \leqslant K$ for each n and $x \in S_\delta[0]$. Define for each r :

$$y_n^r = \begin{cases} \delta^{M/p_k} \operatorname{sgn} b_{nk}, & 0 \leqslant k \leqslant r \\ 0, & r < k \end{cases}$$

where $M = \max(1, \sup_k p_k)$.

Now $y^r \in S_\delta[0]$ and

$$\sum_{k=1}^{r} | b_{nk} | B^{-1/p_k} \leqslant K$$

for each n and r where $B = \delta^{-M}$. Therefore (i) holds. (ii) and (iii) trivially hold.

Sufficiency—Suppose (i)—(iii) hold and $x \in c(p)$. Then there exists l such that $|x_k - l|^{p_k} \to 0$. Hence for $0 < \epsilon < 1$ there exists $k_0 : \forall\, k > k_0$

$$|x_k - l|^{p_k/M} \leqslant \frac{\epsilon}{B(2D+1)} < 1$$

and therefore for $k > k_0$

$$B^{1/p_k} |x_k - l| < B^{M/p_k} |x_k - l| < \left(\frac{\epsilon}{2D+1}\right)^{M/p_k} < \frac{\epsilon}{2D+1}.$$

By (i) and (ii) we have

$$\sum_k |b_{nk} - \alpha_k| B^{-1/p_k} < 2D.$$

Hence

$$\sum_{k>k_0} |(b_{nk} - \alpha_k)(x_k - l)| < \epsilon.$$

Also

$$\sum_{k\leqslant k_0} |(b_{nk} - \alpha_k)(x_k - l)| \to 0 \text{ as } n \to \infty.$$

Therefore we have

$$\lim_{n\to\infty} \sum_k b_{nk} x_k = l\alpha + \sum_k \alpha_k (x_k - l).$$

This completes the proof.

Theorem 2—$A \in (l_\infty(p), C_s)$ if and only if

(i) for all integers $N > 1$

$$\sum_k |b_{nk}| N^{1/p_k} \text{ converges uniformly in } n$$

(ii) $\lim_{n\to\infty} b_{nk} = \alpha_k\ (\forall\, k)$.

Proof : Suppose that $A \in (l_\infty(p), C_s)$. Clearly (ii) holds. If (i) is false then the matrix $C = (C_{nk}) = (a_{nk} N^{1/p_k}) \notin (l_\infty, C_s)$ for some integer $N > 1$, see Stieglitz and Tietz[3]. So there exists $x \in l_\infty$ such that $Cx \notin C_s$. Now $y = (y_k) = (N^{1/p_k} x_k) \in l_\infty(p)$, but $Ay = Cx \notin C_s$ and this contradiction completes the proof.

For the sufficiency, suppose that the conditions hold. Take an integer $N > \max(1, \sup |x_k|^{p_k})$. We have

$$\left|\sum_k (b_{nk} - \alpha_k) x_k\right| < \sum_k |b_{nk} - \alpha_k| N^{1/p_k}.$$

By (i) and (ii) we have

$$\lim_{n\to\infty} \sum_k b_{nk}\, x_k = \sum_k \alpha_k\, x_k.$$

This completes the proofs.

Theorem 3—$A \in (l\,(p), C_S)$ if and only if

(i) there exists an integer $B > 1$ such that

$$C\,(B) > \infty \;(1 < p_k < \infty).$$

$$\sup_{n,k} \mid b_{nk} \mid^{p_k} > \infty \;(0 < p_k \leqslant 1)$$

(ii) $\lim_{n\to\infty} b_{nk} = \alpha_k \;(A\;\; k).$

This is an immediate consequence of the Corollary to Theorem 1 of Lascarides and Maddox[1], if one notices that $A \in (l\,(p), C_S)$ if and only if the matrix $B = (b_{nk}) \in (l\,(p), c)$.

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References

1. C. G. Lascarides and I. J. Maddox, *Proc. Camb. phil. Soc.* **68** (1970), 99–104.
2. I. J. Maddox, *Proc. Camb. philo. Soc.* **68** (1969), 431–35.
3. I. J. Maddox, *Elements of Functional Analysis*. Cambridge University Press, 1970.
4. S. Nanda, *Queen's Papers in Pure and Appl. Math. No.* 74. Queen's University Press, 1986.
5. M. Stieglitz and H. Tietz, *Math. Z.* **154** (1977) 1–16.

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TRANSIENT FORCED AND FREE CONVECTION FLOW PAST AN INFINITE VERTICAL PLATE

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An exact solution to the unsteady free and forced convection flow of an incompressible viscous fluid past an infinite vertical plate is presented and the expressions for the velocity, the penetration distance and the skin-friction are derived. It is observed that an increase in the Prandtl number leads to a decrease in the penetration distance and the skin-friction when the time t is constant.

1. INTRODUCTION

Siegel[1], Schetz and Eichhorn[2], Menold and Yang[3], Chung and Anderson[4], Goldstein and Briggs[5] and Sugawava and Michiyoshi[6] studied the unsteady free convection flow under different condition past an infinite vertical plate. Goldstein and Eckert[7] confirmed experimentally some of these theoretical predictions. In all these studies, the infinite plate was assumed to be stationary and the fluid was supposed to move due to temperature difference only. If the fluid is stationary and the infinite plate, surrounded by the stationary fluid, is given an impulsive motion along with its temperature raised to T_w such that $T_w' > T_\infty$, where T_∞, is the temperature of the surrounding fluid, how the flow of the fluid takes its shape ? This was studied by Soundalgekar[8] in case of an isothermal plate. The effects of free convection currents on the flow and the skin-friction were studied in this paper. However, another physical situation which is often experienced in the industrial application is the unsteady free and forced convective flow past an infinite vertical isothermal plate of an incompressible fluid. This has not been studied in the literature. Hence the motivation to undertake this study. In section 2, the mathematical analysis is presented and in section 3, the conclusions are setout.

2. Mathematical Analysis

Here we consider the unsteady free and forced convection flow of a viscous incompressible fluid past an infinite vertical isothermal plate in the upward direction. The x'-axis is taken along the plate in the vertically upward direction and the y'-axis is taken normal to the plate. Then the physical variables are functions of y' and t' only. Then under usual Boussinesq's approximation, the flow is governed by the system of equations :

$$\rho \frac{\partial u'}{\partial t'} = g \beta \rho (T' - T'_\infty) + \mu \frac{\partial^2 u'}{\partial y'^2} \quad ...(1)$$

$$\rho C_p \frac{\partial T'}{\partial t'} = k \frac{\partial^2 T}{\partial y'^2} \quad ...(2)$$

The initial and boundary conditions are

$$\left.\begin{array}{l} t' \leqslant 0, u' = 0, T' \to T'_\infty, \text{ for all } y' \\ t' > 0, u' = 0, T \to T'_w \text{ at } y' = 0. \\ u' \to U_0, T' \to T'_\infty \text{ as } y' \to \infty. \end{array}\right\} \quad ...(3)$$

Here u' is the velocity of the fluid in the x'-direction, ρ' the density, g the acceleration due to gravity, β the coefficient of volume expansion, T' the temperature of the fluid near the plate, T'_∞ the temperature of the fluid in the free-stream, μ the coefficient of viscosity, C_p the specific heat at constant pressure and k is the thermal conductivity. Initially, the plate temperature and the free stream temperature are the same everywhere. At $t' > 0$, the plate temperature T'_∞ is raised to T'_w .

On introducing the following non-dimensional quantities

$$\left.\begin{array}{l} y = y' U_0 Gr^{1/2}/\nu, t = t' U_0^2 Gr/\nu, u = u'/U_0 \\ Pr = \mu C_p/k, \theta = (T' - T'_\infty)/(T'_w - T'_\infty), \\ Gr = \nu g \beta (T'_w - T'_\infty)/U_0^3 \text{ (the Grashof number)} \end{array}\right\} \quad ...(4)$$

in eqns. (1) – (3), we have

$$\frac{\partial u}{\partial t} = \theta + \frac{\partial^2 u}{\partial y^2} \quad ...(5)$$

$$Pr \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial y^2} \quad ...(6)$$

and the initial and boundary conditions are

$$\left.\begin{array}{l} t \leqslant 0,\ u = 0,\ \theta = 0, \text{ for all } y \\ t \geqslant 0,\ u = 0,\ \theta = 1, \text{ at } y = 0 \\ u = 1,\ \theta = 0 \text{ as } y \to \infty. \end{array}\right\} \quad \text{...(7)}$$

Equations (5) - (6) subject to the conditions (7) are solved by the usual Laplace-transform technique and the solutions are as follows :

$$\begin{aligned} u = 1 - \operatorname{erfc}(\eta) - \frac{t}{Pr - 1} & [\operatorname{erfc}(\eta \sqrt{Pr}) - \operatorname{erfc}(\eta)] \\ & + 2\eta^2 (Pr \operatorname{erfc}(\eta \sqrt{Pr}) - \operatorname{erfc}(\eta)) \\ & + \frac{2\eta}{\sqrt{\pi}} (\exp(-\eta^2) - \sqrt{Pr} \exp(-Pr.\eta^2) \end{aligned} \quad \text{...(8)}$$

$$\theta = \operatorname{erfc}(\eta \sqrt{Pr}) \quad \text{...(9)}$$

knowing the velocity field, it is more interesting to study the leading edge effect. The penetration distance is derived by integrating u with respect to t and the maximum penetration distance $x_{p_{max}}$ at any time can be determined by differentiating x_p with respect to y holding t constant and then by setting the derivative equal to zero. Thus the penetration distance is given by

$$x_p = \int_0^t u(y, t)\, dt. \quad \text{...(10)}$$

This can be expressed in terms of the Laplace transform and inverse transform with respect to the variable t as

$$\begin{aligned} x_p &= L^{-1} \left\{ \frac{1}{s} L(u(y, t)) \right\} \\ &= L^{-1} \left\{ \frac{1}{s} \bar{u}(y, s) \right\}. \end{aligned} \quad \text{...(11)}$$

Substituting for $\bar{u}(y, s)$ in (11), and taking the inverse, we have

$$\begin{aligned} X_p = t & \left[(1 - \operatorname{erfc}(\eta)) + \eta \left(\frac{2 \exp(-\eta^2)}{\sqrt{\pi}} - 2\eta \operatorname{erfc}(\eta) \right) \right] \\ & - \frac{t^2}{6(Pr - 1)} \Big[4\eta^2 \{(\eta^2 P_r^2 + 3\, Pr) \operatorname{erfc}(\eta \sqrt{Pr}) \\ & - (\eta^2 + 3) \operatorname{erfc}(\eta)\} + 3 (\operatorname{erfc}(\eta \sqrt{Pr}) - \operatorname{erfc}(\eta)) \\ & + \frac{2\eta}{\sqrt{\pi}} ((2\eta^2 + 5) \exp(-\eta^2) - (2Pr\, \eta^2 + 5) \sqrt{Pr} \exp(-Pr\, \eta^2) \Big]. \end{aligned} \quad \text{...(12)}$$

To understand the physical meaning of the problem, we have calculated the numerical values of x_p for different values of Pr and these are plotted in Fig. 1. We observe from this Fig. 1 that the penetration distance x_p decreases with increasing the Prandtl number of the fluid. But it increases with t, the time.

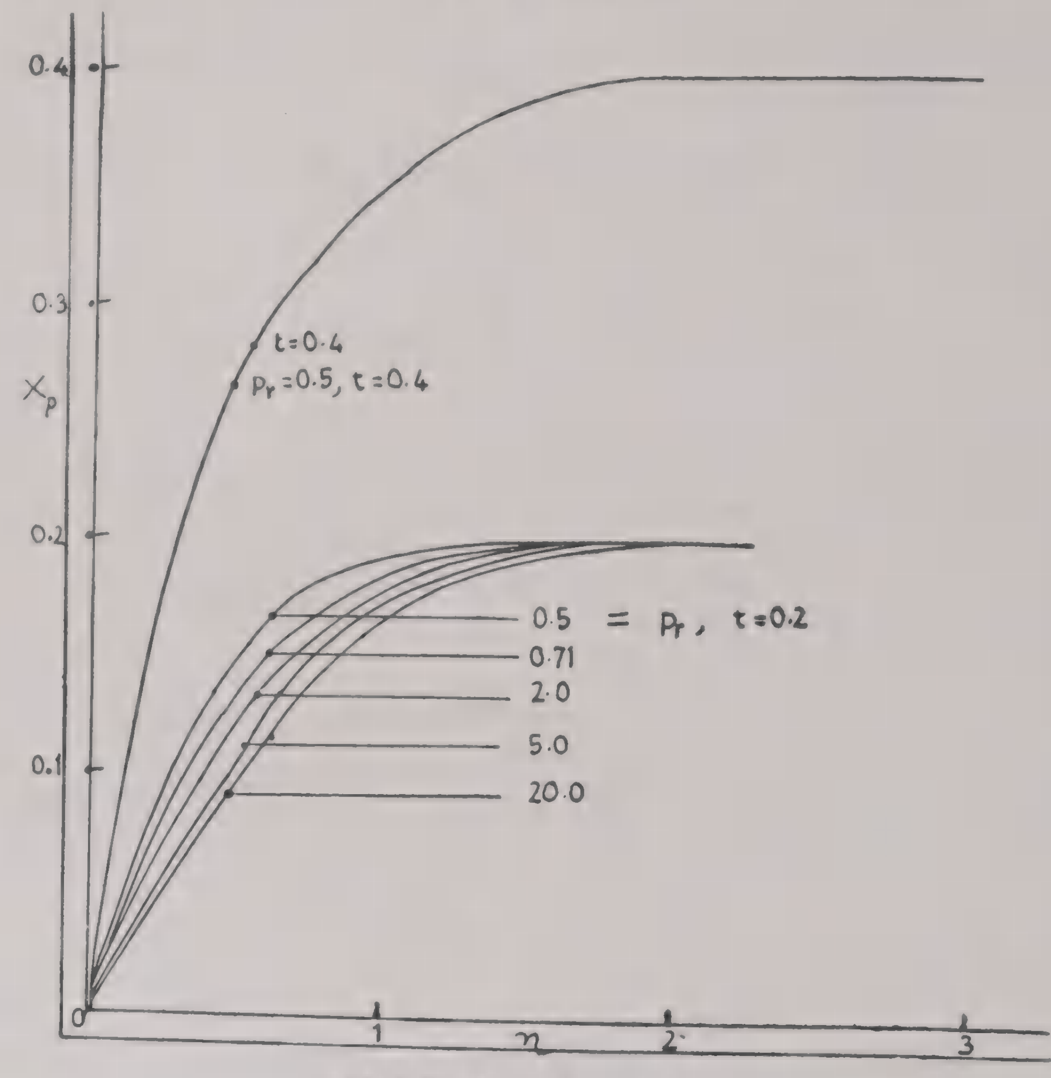

FIG. 1 Penetration distance

We now study the skin-friction. It is given by

$$\tau' = -\mu \left. \frac{\partial u'}{\partial y'} \right|_{y'=0} \qquad \ldots(13)$$

and in view of (4), equation (13) reduces to

$$\tau = -\tau'/\rho U_0^2 \; Gr^{1/2} = \left. \frac{du}{d\eta} \right|_{\eta=0}. \qquad \ldots(14)$$

Hence substituting for u from (12) in equation (14) and simplifying, we get,

$$\tau = \frac{1}{\sqrt{\pi t}} + \frac{2\sqrt{t}}{\sqrt{\pi}} \cdot \frac{1}{1+Pr} . \qquad \ldots(15)$$

From (15), we conclude that the skin-friction decreases with increasing Pr when t is constant.

3. Conclusion

The penetration distance and the skin-friction decrease with increasing the Prandtl number.

References

1. R. Seigel, *Trans. Am. Soc. Mech. Engnrs.* **80** (1958), 347–59.
2. J. A. Schetz and R. Eichhorn, *J. Heat Transfer (Trans. ASME)* **84C** (1962), 334–38.
3. E. R. Menold and K. T. Young, *J. Appl. Mech. (Trans. ASME)* **29E** (1962), 124–26.
4. P. M. Chung and A. D. Anderson, *J. Heat Transfer (Trans. ASME)* **83C** (1961), 473–78.
5. R. S. Goldstein and D. G. Briggs, *J. Heat Transfer (Trans. ASME)* **86C** (1964), 490–500.
6. S. Sugawara and I. Michiyoshi, *Proc. of the 1st Japan Congress of Applied Mechanics*, (1951), 501–506.
7. R. J. Goldstein and E. R. G. Eckert, *Int. J. Heat Mass Transfer* **1** (1960), 208–18.
8. V. M. Soundalgekar, *J. Heat Transfer (Trans. ASME)* **99C** (1977), 499–501.

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EFFECT OF THERMAL DIFFUSION ON THERMOHALINE INTERLEAVING IN A POROUS MEDIUM DUE TO HORIZONTAL GRADIENTS

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The effect of thermal diffusion on the onset of instability in an unbounded vertically stratified, stable, quiescent theromohaline fluid with compensating horizontal thermal and salinity gradients saturating a porous medium is investigated. It is found that for this system, instability sets in only through stationary mode when the wavelength is large compared to the porosity of the medium. The dependance on thermal diffusion of (a) the maximum growth rate (b) the ratio of fluxes (c) the onset and region of instability, are shown using the Soret parameter, S. For (i) $S = \tau^{-1} - 1$, the system remains stable for small perturbations (ii) $S < -1$, convection sets in even in the absence of the horizontal gradient when both temperature and salinity gradients are stable where τ is the ratio of mass diffusivity to thermal diffusivity. It is also shown that salt gets transported faster than heat when $S < \tau^{-1} - 1$ while transport of heat is more for $S > \tau^{-1} - 1$ for some growth rate of instabiiity.

1. INTRODUCTION

The study of thermohaline instability with thermal diffusion in a fluid saturated porous medium is of importance in geophysics, ground water hydrology, soil sciences, oil extraction, extraction of ores etc., because it is known that the earth's crust is a porous medium saturated by a mixture of different types of fluids like oil, water, gases and molten form of ores or ores dissolved in fluids. Thermal gradient present between the interior and exterior of the earth's crust may help convection to set in. Also the two transport processes (heat and mass transfer) interfere with each other and produce cross phenomena known as thermal diffusion (Soret effect) and diffusion thermo (Dufour effect)[1,2]. Thermal diffusion is the flux of mass caused by a temperature gradient and diffusion thermo is the flux of heat caused by a concentration gradient. But,

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in general, in liquid mixtures diffusion thermo is negligible for a heat-solute pair. Various authors have analysed Soret driven instability for Newtonian fluid layers[3-5] as well as for fluid saturated porous media[6-14].

All these authors have ignored the horizontal gradients of temperature and salinity that may be present. The earth's crust made up of large sedimentary materials of different permeabilities and heat capacities[15] causes nonuniform heating of the lower layers resulting in a horizontal temperature gradient. Also, due to the nonuniform distribution of the constituent solutes in the multicomponent system saturating the earth's crust, one encounters horizontal solutal gradients in addition to the vertical ones.

The aim of the present paper is to study the effects of both thermal diffusion and horizontal gradients on thermohaline instability in a fluid-saturated porous medium. Holyer[16] has analysed double-diffusive (thermohaline) interleaving due to horizontal gradients in a Newtonian fluid layer. The present work is based on Holyer's[16] work.

Numerical computations are made by choosing the values of the Soret parameter S at random since it is reported by Hurle and Jakeman[3], Schechter *et al.*[4], and Legros *et al*[18], that the sign and magnitude of the Soret coefficient depend on the velocity of flow and/or solute concentration of the system.

2. Mathematical Formulation

Consider an unbounded region of a porous medium saturated with a Boussinesq fluid which is quiescent and which has both temperature and salinity variations. It is assumed that the horizontal temperature and salinity gradients compensate each other in the basic state and hence there is no change in the density in the horizontal direction. The mass flow generated by the temperature gradient is taken into account in the formulation of the problem which is analysed for two-dimensional perturbations. The coordinate axes are chosen with z-axis vertically upwards and x-axis horizontal in the direction of increasing salinity. Only a stable basic state (i. e. where the total density gradient is negative) is considered to study the problem. The governing equations are

$$\frac{\partial u}{\rho x} + \frac{\partial \omega}{\partial z} = 0 \qquad \ldots(1)$$

$$\frac{1}{\phi} \frac{\partial \vec{q}}{\partial t} + A \mid \vec{q} \mid \vec{q} = \frac{-\nabla p}{\rho_0} + \frac{\rho \vec{g}}{\rho_0} - \frac{\nu \vec{q}}{k} \qquad \ldots(2)$$

where A is a constant dependent on the porosity and the geometrical parameters of the porous medium. This is the Navier-Stokes equation modified for Boussinesq approximation and Darcy law[9,17]:

$$\frac{\partial T}{\partial t} + M \vec{q} . \nabla T = K \nabla^2 T \qquad \ldots(3)$$

$$\frac{\partial C}{\partial t} + \vec{q}.\ \nabla C = D\ \nabla^2 C + D_1\ \nabla^2 T \qquad \text{...(4)}$$

$$\rho = \rho_0\,[1 - \alpha_T\,(T - T_0) + \alpha_c\,(C - C_0)] \qquad \text{...(5)}$$

where u, w are the horizontal and the vertical components of the velocity $\vec{q}$: t, p, ν, ρ are the time, pressure, kinematic viscosity and density of the double diffusive fluid; ϕ and k are the porosity and permeability of the medium; $\vec{g}$ is the acceleration due to gravity; T and C are the temperature and concentration; ρ_0 is the density at temperature T_0 and concentration C_0; K and D are the thermal and solutal diffusivities; D_1 is the Soret coefficient; α_T and α_c are the coefficients of thermal and mass expansions which are both positive; $M = (\rho_0\ c_p)_f/(\rho_0\ c_p)_m$, c_p is the heat capacity, suffixes f and m stand for fluid and fluid-solid mixtare. But M has been dropped in the subsequent analysis because liquid-saturated porous media is considered for this work.

Small perturbations are imposed on the basic state given by

$$\vec{q}_b = 0;\ T_b\,(x, z) = \bar{T}_x\, x + \bar{T}_z\, z + T_0;\ C_b\,(x, z) = \bar{C}_x\, x + \bar{C}_z\, z + C_0;$$

$$\frac{\partial \rho_b}{\partial x} = 0 \text{ (by assumption) i. e., } \alpha_T\,\bar{T}_x - \alpha_c\,\bar{C}_x = 0$$

where $\vec{q}_b$, $T_b\,(x, z)$, $C_b\,(x, z)$ are the velocity, temperature and concentration in the basic state; $\bar{T}_x$, $\bar{T}_z$ are the horizontal and vertical temperature gradients; $\bar{C}_x$, $\bar{C}_z$ are the horizontal and vertical salinity gradients. The perturbed state is given by

$$T_b\,(x, z) + T\,(x, z, t);\ C_b\,(x, z) + C\,(x, z, t);$$

$$p_b\,(z) + p\,(x, z, t);\ \vec{q}\,(x, z, t)$$

where p_b is independent of x from the basic state momentum equation.

Let the stream function ψ be defined by $u = -\dfrac{\partial \psi}{\partial z}$ and $w = \dfrac{\partial \psi}{\partial x}$.

The linearised equations in T, C and ψ are given by

$$\left(\frac{\partial}{\partial t} - K\,\nabla^2\right) T = \frac{\partial \psi}{\partial z}\,\bar{T}_x - \frac{\partial \psi}{\partial x}\,\bar{T}_z \qquad \text{...(6)}$$

$$\left(\frac{\partial}{\partial t} - D\,\nabla^2\right) C = D_1\,\nabla^2 T + \frac{\partial \psi}{\partial z}\,\bar{C}_x - \frac{\partial \psi}{\partial x}\,\bar{C}_z \qquad \text{...(7)}$$

$$\left(\frac{1}{\phi}\,\frac{\partial}{\partial t} + \frac{\nu}{k}\right) \nabla^2 \psi = g\left(\alpha_T\,\frac{\partial T}{\partial x} - \alpha_c\,\frac{\partial C}{\partial x}\right). \qquad \text{...(8)}$$

On eliminating T and C (6) — (8) lead to

$$\left(\frac{\partial}{\partial t} - K\,\nabla^2\right)\left(\frac{\partial}{\partial t} - D\,\nabla^2\right)\left(\frac{1}{\phi}\,\frac{\partial}{\partial t} + \frac{\nu}{k}\right)\nabla^2 \psi$$

(*equation continued on p.* 719)

$$= - g\,\alpha_T\,\bar{T}_z \left(\frac{\partial}{\partial t} - D\,\nabla^2 \right) \frac{\partial^2 \psi}{\partial x^2} + g\,\alpha_c\,\bar{C}_z \left(\frac{\partial}{\partial t} - K\,\nabla^2 \right) \frac{\partial^2}{\partial x^2}$$

$$+ g\,\alpha_c\,\bar{C}_x\,(K - D)\,\nabla^2\,\psi_{zx}$$

$$+ g\,\alpha_c\,D_1 \left(\bar{T}_z \frac{\partial^2}{\partial x^2} - \bar{T}_x \frac{\partial^2}{\partial z\,\partial x} \right) \nabla^2\,\psi. \qquad \ldots(9)$$

The solution to (9) is assumed in the form

$$\psi = \psi_0 \exp\,(i\,a_x\,x + i\,a_z\,z + \sigma\,t).$$

On substitution for ψ in (9), the equation in σ which decides the stability of the system is given by

$$(\sigma + K\,a^2)\,(\sigma + D\,a^2) \left(\sigma\,\frac{a^2}{\phi} + \frac{\nu\,a^2}{k} \right)$$

$$+ \sigma\,g\,a_x^2\,(\alpha_T\,\bar{T}_z - \alpha_c\,\bar{C}_z) + g a^2\,\alpha_c\,\bar{C}_x\,[K - D\,(1 + S)]\,a_x\,a_z$$

$$- g a^2\,[K\,\alpha_c\,\bar{C}_z - D\,(1 + S)\,\alpha_T\,\bar{T}_z]\;a_x^2 = 0$$

where a_x and a_z are the horizontal and vertical wave numbers and $a^2 = a_x^2 + a_z^2$; $S = \alpha_c\,D_1/\alpha_T\,D$ is the Soret parameter.

For $a^2/\phi \ll 1$ (i. e., when the wave number is small compared to ϕ or, the wave-length is large when compared to ϕ. This does not imply that ϕ is large. It is small enough to justify Darcy model with 'a' $\ll \phi$), the above equation becomes

$$(\sigma + K\,a^2)\,(\sigma + D\,a^2)\;\frac{\nu\,a^2}{k} + \sigma\,g\,a_x^2\,(\alpha_T\,\bar{T}_z - \alpha_c\,\bar{C}_z)$$

$$+ g\,a^2\,\alpha_c\,\bar{C}_x\,[K - D\,(1 + S)]\,a_x\,a_z$$

$$- g\,a^2\,[K\,\alpha_c\,\bar{C}_z - D\,(1 + S)\;\alpha_T\;\bar{T}_z]\,a_x^2 = 0. \qquad \ldots(10)$$

For neutral stability through stationary mode (i. e., $\sigma = 0$) (10) gives

$$\frac{K\,D\;\nu\,a^4}{k} = - g\,\alpha_c\,\bar{C}_x\,[K - D\,(1 + S)]\,a_x\,a_z$$

$$+ g\,[K\,\alpha_c\,\bar{C}_z - D\,(1 + S)\,\alpha_T\,\bar{T}_z]\,a_x^2\,. \qquad \ldots(11)$$

From (11) it is seen that 'a' will be real when

$$\alpha_c\,\bar{C}_x\,[1 - \tau\,(1 + S)]\,\frac{a_z}{a_x} < \alpha_c\,\bar{C}_z - \tau\,(1 + S)\,\alpha_T\,\bar{T}_z$$

where $\tau = \dfrac{D}{K}$. $\qquad \ldots(12)$

The inequality (12) is the same for a Newtonian fluid layer (i. e., without porous medium) and hence the values of a_z/a_x for which convection sets in through stationary mode in a Newtonian fluid layer are the same even in the presence of a porous medium.

For neutral stability through oscillatory mode ($c = i\omega$, where ω real is the frequency) the equation (10) gives

$$
\begin{aligned}
& - \frac{\omega^2 a^2 \nu}{k} + \frac{K D \nu a^6}{k} + g a^2 \alpha_c \bar{C}_x [K - D(1 + S)] a_x a_z \\
& - g a^2 [K \alpha_c \bar{C}_z - D(1 + S) \alpha_T \bar{T}_z] a_x^2 + i\omega \left[\frac{a^4}{k} (K + D) \nu \right. \\
& \left. + g a_x^2 (\alpha_T \bar{T}_z - \alpha_c \bar{C}_z) \right] = 0. \qquad (13)
\end{aligned}
$$

The imaginary part of (13) equated to zero gives

$$
\omega \left[\frac{a^4}{k} (K + D)\nu + g a_x^2 (\alpha_T \bar{T}_z - \alpha_c \bar{C}_z) \right] = 0.
$$

implying $\omega = 0$ for a stable basic state $(\alpha_T \bar{T}_z - \alpha_c \bar{C}_z > 0)$.

Hence, when the wavelength is large compared to ϕ convection does not set in through oscilatory mode. Under this assumption, only stationary convection is discussed in what follows. Since the Soret perameter S can have any value[3,4,18] the discussion is based on different ranges of S

The basic state in which both vertical thermal and salinity gradients are stable ($\alpha_T \bar{T}_z > 0$, $\alpha_c \bar{C}_z < 0$) is considcred in the discussion given below. It is seen from (12) that the inequality cannot be satisfied when $\bar{C}_x = 0$ for $S \geqslant -1$. This shows that $\bar{C}_x$ derives the instability and if, in addition, $S < \tau^{-1} - 1$, the inequality (12) gives $\frac{a_z}{a_x} < 0$ implying that hot salty solution overlies cold fresh solution which is usually the set up for stationary mode. But for $S < -1$, convection can set even in the absence of $\bar{C}_x$ provided $\alpha_c \bar{C}_z - \tau(1 + S) \alpha_T \bar{T}_z > 0$. A difference in the set up with $\bar{C}_x \neq 0$ is noticed for $S > \tau^{-1} - 1$, when the stationary convection sets in with $\frac{a_z}{a_x} > 0$ for which cold fresh solution is above hot salty solution which is a diffusive regime. This tilting has been reported by McDougall[19] for a Newtonian fluid layer. It is also seen that (12) is not satisfied for any $\bar{C}_x$ when $S = \tau^{-1} - 1$, i. e., a system which is stable in the basic state remains stable for small perturbations, implying that $\bar{C}_x$ is compensated by this value of the Soret parameter. A similar analysis can be carried out for other directions of the gradients subject to a stable basic state condition. The regions of stationary instability in the $\frac{a_z}{a_x} - R_t$ plane are shown in Fig. 1 (a) — (c) for different values of S and H for given R_c

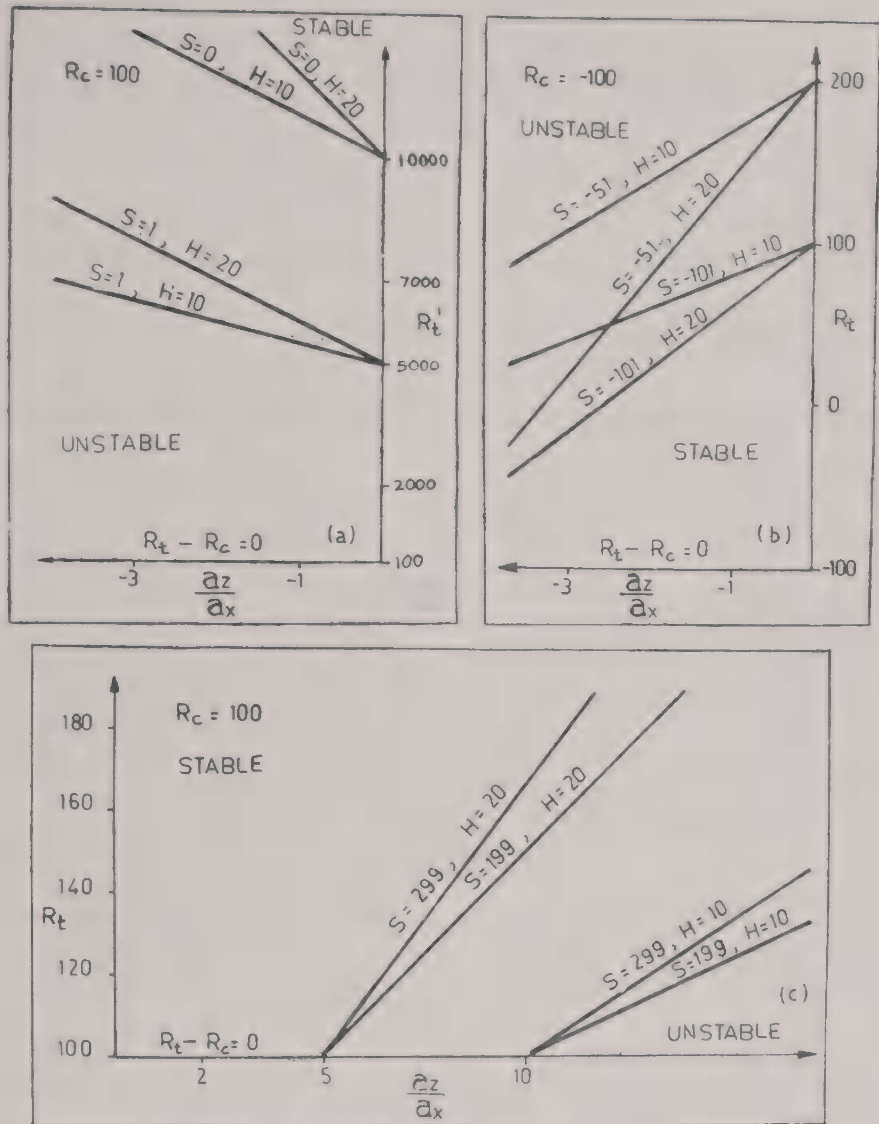


Fig. 1. $\frac{a_z}{a_x} - R_t$ plots for $\tau = 0.01$ and different values of H.

(a) $-1 < S < \tau^{-1} - 1$. The region of instability lies between the line considered and $R_t - R_c = 0$.

(b) $S < -1$. The region of instability is above the line considered.

(c) $S > \tau^{-1} - 1$. The region of instability lies between the line considered and $R_t - R_c = 0$.

where the horizontal Rayleigh number $H = \dfrac{g\, \alpha_c\, \overline{C}_x\, k}{\nu\, K\, a^2}$;

the vertical thermal Rayleigh number $R_t = \dfrac{g\, \alpha_T\, \overline{T}_z\, k}{\nu\, K\, a^2}$;

the vertical salinity Rayleigh number $R_c = \dfrac{g\, \alpha_c\, \overline{C}_z\, k}{\nu K\, a^2}$.

It can be seen from these figures that an increase in H increases the region of instability. Figure 1 (a) shows that when S increases, the region of instability decreases. Figures 1(b)—(c) also reveal that the Soret parameter affects the regions of instability.

3. Maximum Growth Rate of Instability

The growth rate of unstable mode is given by σ which has to be real for the instability to grow via stationary mode. The maximum growth is obtained from

$$\frac{\partial \sigma}{\partial a_x} = 0 = \frac{\partial \sigma}{\partial a_z}. \quad \text{...(14)}$$

Proeeeding on the same lines as Holyer[16], the Rayleigh numbers for the maximum growth rate are given by

$$H = -2 \frac{a_z}{a_x} \frac{(\lambda + 1)(\lambda + \tau)}{[1 - \tau(1 + S)]} \quad \text{...(15a)}$$

$$R_t = \frac{(\lambda + 1)[a^2 \tau (\lambda + 1) - 2a_z^2 \lambda(\lambda + \tau)]}{\lambda a_x^2 [1 - \tau(1 + S)]} \quad \text{...(15b)}$$

$$R_c = \frac{1}{\lambda a_x^2 [1 - \tau(1 + S)]} \{a^2 [\lambda^2 (1 - \tau S) + 2\lambda\tau + \tau^2 (1 + S)] - 2a_z^2 \lambda(\lambda + 1)(\lambda + \tau)\} \quad \text{...(15c)}$$

where

$$\lambda = \frac{\sigma}{K a^2}.$$

It can be seen that S affects all quantities though $\frac{H}{R_t}$ is independent of S. To find the maximum dimensionless growth rate λ and the corresponding horizontal and vertical wave numbers a_x, c_z, equations (15) are to be slmultaneously solved for given and values of the gradients and τ. But, following Baines and Gill[20] and Holyer[16], it is easier to compute H, R_t, R_c assigning values for $\frac{a_z}{a_x}$ and λ for given τ. $\frac{H}{R_t} - \frac{R_c}{R_t}$ plots are displayed in Fig. 2 (a), (b) and 3 for different values of S for given values of λ and $\frac{a_z}{a_x}$. These figures are drawn for positive values of R_t subject to ths condition $R_t - R_c > 0$ (stable basic state) and for values of S in the range $-1 \leqslant S < \tau^{-1} - 1$ with $\tau = 0.01$ for a heat-salt pair. It is seen from Figures 2(a), (b) and 3 that when S increases $\frac{R_c}{R_t}$ increases. The effect of S becomes less for higher growth rates. For the growth rate $\lambda = \tau^{1/2}$, it is found that $\frac{R_c}{R_t} = 1$ for all $\frac{a_z}{a_x}$ and S, i. e., $\frac{H}{R_t}$

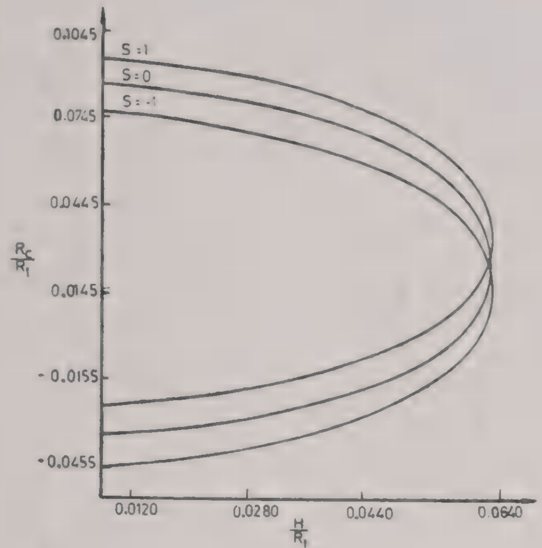


FIG. 2. (a) $\frac{H}{R_t} - \frac{R_c}{R_t}$ plots for $R_t > 0$ subject to $R_t - R_c > 0$ for $\tau = 0.01$, $\lambda = 0.02$, $-1 \leqslant S < \tau^{-1} - 1$.

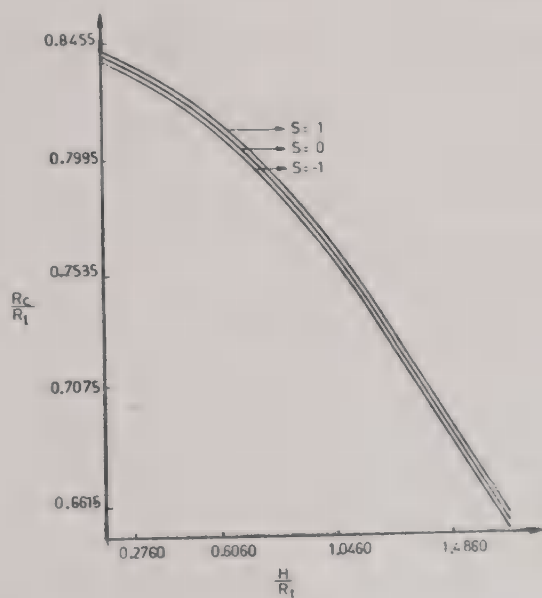


FIG. 2 (b) $\frac{H}{R_t} - \frac{R_c}{R_t}$ plots for $R_t > 0$ subject to $R_t - R_c > 0$ for $\tau = 0.01$, $\lambda = 0.09$, $-1 \leqslant S < \tau^{-1} - 1$.

$-\frac{R_e}{R_t}$ plot is the same straight line (not shown in the figure). Figure 3 shows that $\frac{R_c}{R_t}$ decreases when $\left| \frac{a_z}{a_x} \right|$ increases and the effect of S is very nearly the same for different $\frac{a_z}{a_x}$.

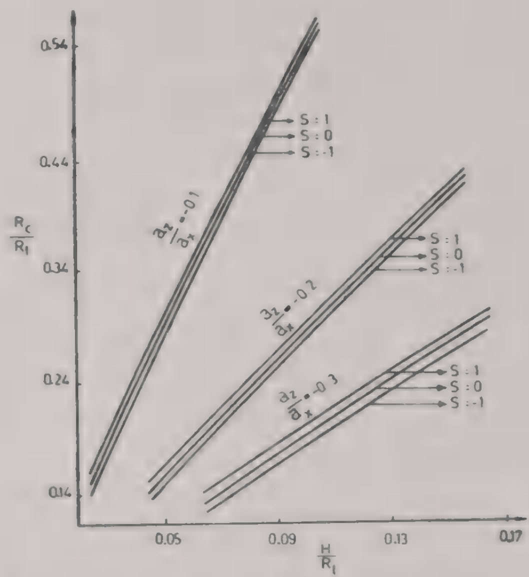


FIG. 3. $\frac{H}{R_t} - \frac{R_c}{R_t}$ plots for $R_t > 0$ subject to $R_t - R_c > 0$ for $\tau = 0.01$, $\frac{a_z}{a_x} = -0.1$ -0.2, -0.3 and $- < S < \tau^{-1} - 1$.

For marginal stability for which $\lambda \to 0$, it is easily seen from (15) that

$$\frac{H}{R_t} \to 0, \quad \frac{Rc}{Rt} \to \tau(1 + S) \text{ with } R_t \to \infty$$

hence $a \to 0$. When $\lambda \to \infty$, it is seen that $\frac{H}{R_t} \to \frac{a_x}{a_z}$ and $\frac{Rc}{R_t} \to 1$. These results hold good in the absence of porous medium also.

4. FLUXES

For a propagating mode,

$$(\psi, T, C) = \text{Re}\,[(\psi_0, T_0, C_0) \exp\,(i\,(a_x\, x + i\, a_z\, z + \sigma\, t)] \quad \text{... (16)}$$

where ψ_0 is assumed and real and σ is positive for a stationary mode. Rewriting equations (3) (for $M = 1$ and (4) in the form

$$\frac{\partial T}{\partial t} + \psi_x \bar{T}_z - \psi_z \bar{T}_x = K\, \nabla^2 T$$

$$\frac{\partial C}{\partial t} + \psi_x \bar{C}_z - \psi_z \bar{C}_x = D\, \nabla^2 C + D_1\, \nabla^2 T$$

and substituting (16), T_0 and C_0 are given by

$$T_0 = \frac{i\, \psi_0\, (a_z\, \bar{T}_x - a_x\, \bar{T}_z)}{\sigma + K\, a^2} \quad \text{...(17a)}$$

$$C_0 = i\, \psi_0 \left[\frac{a_z\, \bar{C}_x - a_x\, \bar{C}_z}{\sigma + D\, a^2} \right.$$

(*equation continued on p.* 725)

$$- D_1 a^2 \frac{(a_z \bar{T}_x - a_x \bar{T}_z)}{\sigma + K a^2} \Bigg] . \qquad \ldots(17b)$$

The vertical heat flux F_T and vertical salt flux F_C are given by

$$F_T = \tfrac{1}{2} \frac{a_z \bar{T}_x - a_x \bar{T}_z}{\sigma + K a^2} a_x \psi_0^2$$

$$F_C = \tfrac{1}{2}\left[\frac{a_z \bar{C}_x - a_x \bar{C}_z}{\sigma + D a^2} - \frac{D_1 a^2 (a_z \bar{T}_x - a_x \bar{T}_z)}{(\sigma + D a^2)(\sigma + K a^2)} \right] a_x \psi_0^2$$

where $F_T = < \omega T >$, $F_C = <\omega C>$ and $<>$ denotes an average over a wavelength.

The flux ratio is given by

$$\frac{\alpha_T F_T}{\alpha_c F_C} = \frac{(R_t - \frac{a_z}{a_x} H)(\lambda + \tau)}{(R_c - \frac{a_z}{a_x} H)(\lambda + 1) - S \tau (R_t - \frac{a_z}{a_x} H)} .$$

Substituting for R_t, R_c and H from (15), this becomes

$$\frac{\alpha_T F_T}{\alpha_c F_C} = \frac{\lambda + 1) \tau}{\lambda (1 - \tau S) + \tau} . \qquad \ldots(18)$$

For $S < \tau^{-1} - 1$, it is clear that the flux ratio given by (18) is less than unity for all values of λ and hence salt is transported faster than heat. The flux ratio is

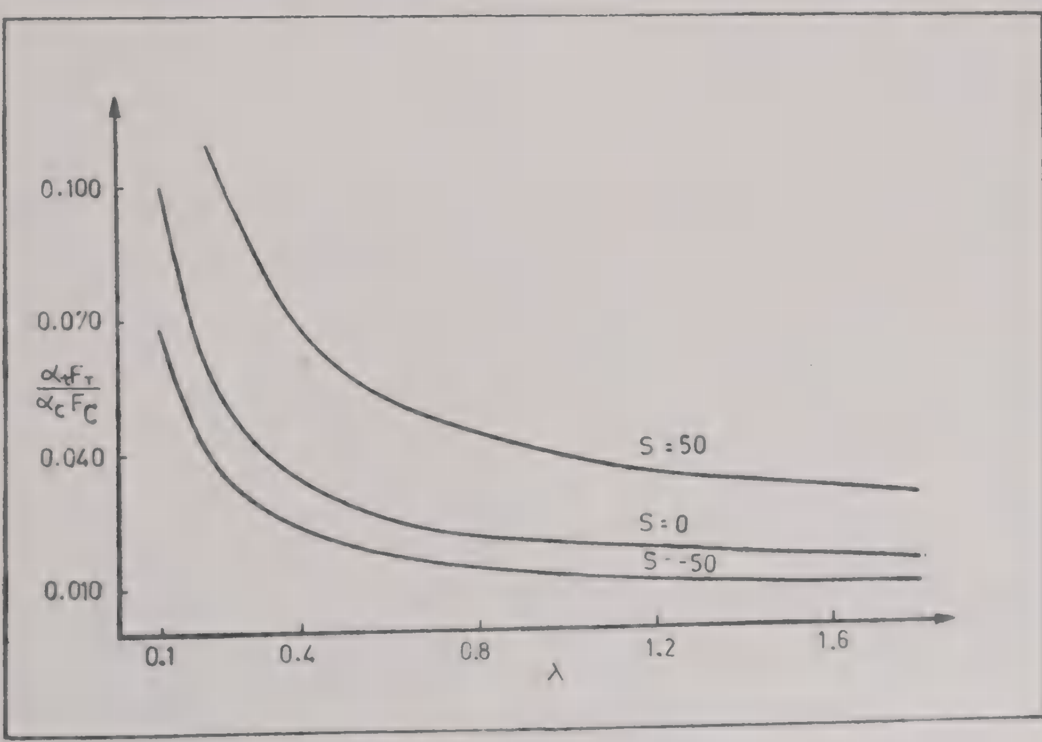


FIG. 4. (a) $\lambda - \frac{\alpha_T F_T}{\alpha_c F_C}$ plots for $\tau = 0.01$ for $S < \tau^{-1} - 1$.

greater than unity (i) for all values of λ when $\tau^{-1} - 1 < S \leqslant \tau^{-1}$ (ii) for $\lambda < \dfrac{\tau}{\tau S - 1}$ wnen $S > \tau^{-1}$ implying that the heat gets transported faster than salt. It is further noticed that for $S > \tau^{-1}$ with $\lambda > \dfrac{\tau}{\tau S - 1}$, transports of heat and salt are in opposite directions. At $\lambda = \dfrac{\tau}{\tau S - 1}$, the flux ratio tends to infinity, i. e., the heat flux is very large when compared to the salt flux. Figures (4) a — (*c*) are $\lambda - \dfrac{\alpha_T F_T}{\alpha_C F_C}$ plots in which the above results are shown. Figure 4 (a) shows that the flux ratio increases when S increases, but decreases when λ increases where $S < \tau^{-1} - 1$. For $S = \tau^{-1} - 1$ there is no convection. It is seen from Fig. 4(b) that the flux ratio is greater than unity for all λ and that it increases with both S and λ when $\tau^{-1} - 1$

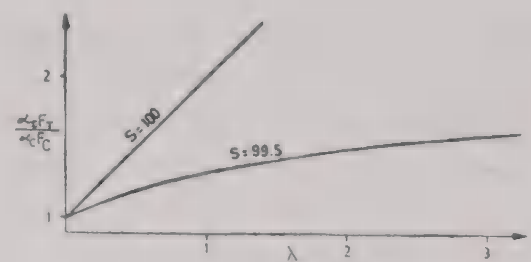
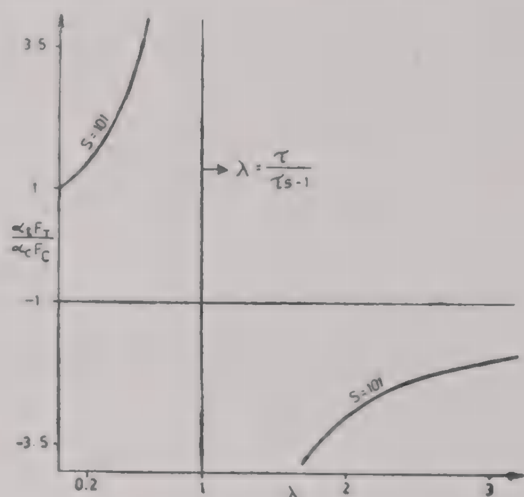


FIG. 4. (b) $\lambda - \dfrac{\alpha_T F_T}{\alpha_c F_C}$ plots for $\tau = 0.01$ for $\tau^{-1} - 1 < S \leqslant \tau^{-1}$.



(c) $\lambda - \dfrac{\alpha_T F_T}{\alpha_b F_C}$ plot for $\tau = 0.01$ for $S > \tau^{-1}$.

$< S \leqslant \tau^{-1}$. The discontinuity of the graph at $\lambda = \dfrac{\tau}{\tau S - 1}$ for $S > \tau^{-1}$ can be noticed from Fig. 4 (c).

5. Conclusion

This paper describes in detail the results which are due to the presence of the horizontal gradient and thermal diffusion in a fluid saturated porous mndium. It is shown that when the vertical gradients of both temperature and salinity are stable (i) for $S \geqslant -1$, there is no convection in the absence of $\bar{C}_x$, but for $S < -1$, conditional convection sets in. (ii) for $-1 \leqslant S < \tau^{-1} - 1$, when $\bar{C}_x$ derives convection, $\dfrac{a_z}{a_x} < 0$ i. e. hot salty solution is above cold fresh solution. (iii) for $S > \tau^{-1} - 1$, with

$\bar{C}_x$ driving stationary convection $\frac{a_z}{a_x}$ is positive i. e. cold fresh solution overlies hot salty solution which is usually a diffusive regime. It is also shown that for $S = \tau^{-1} - 1$, no convection is possible for all $\bar{C}_x$. Also salt gets transported faster than heat when $S < \tau^{-1} - 1$ while transport of heat is more for $S > \tau^{-1}$ for some values of the growth rate.

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References

1. S. R. DeGroot and P. Mazur, *Nonequilibrium Thermodynamics*. North-Holland, Amsterdam, 1962.
2. D. D. Fitts, *Nonequilibrium Thermodynamics*. McGraw-Hill Book Company, New York, 1962.
3. D. T. J. Hurle and E. Jakeman, *J. Fluid Mech.* **47** (1971), 667.
4. R. S. Schechter, I Prigogine and J. R. Hamm, *Phys. Fluids* **15** (1972), 379.
5. M. G. Velarde and R. S. Schechter, *Phys. Fluids* **15** (1972), 1707.
6. N. Rudraiah and P. R. Patil, *5th International Heat Transfer Conference, Tokyo, CT* 3.1 (1974), 79.
7. P. R. Patil and N. Rudraiah, *Int. J. Engng. Sci.* **18** (1980), 1055.
8. P. R. Patil and G. Vaidyanathan, *10th NC FMFP India C4* (1981), 144.
9. D. D. Joseph, D. A. Nield and G. Papanicolaou, *Water Resour. Res.* **18** (1982), 1049.
10. N. Rudraiah and M. S. Malashetty, *ASME J. Heat Transfer* **108** (1986), 872.
11. C. P. Parvathy, P. R. Patil and G. Vaidyanathan, *Reg. J. Energy Heat Mass Transfer* **8** (1986), 153.
12. H. Brand and V. Steinberg, *Physica A* 119 (1983), 327.
13. H. Brand and V. Steinberg, *Phys. Letters* **93A** (1983), 333.
14. M. E. Taslim and V. Narusawa, *ASME J, Heat Transfer* **108** (1985), 221.
15. R. J. M. Dewiest, *Flow Through Porous Media*. Academic, New York, 1969.
16. J. Y. Holyer, *J. Fluid Mech.* 137 (1983), 347.
17. D. D. Joseph, *Stability of Fluid Motions II*. Springer-Verlag, 1976, 54.
18. J. C. Legros, P. Goemaere and J. K. Platten, *Phys. Rev.* **A32** (1985), 1903.
19. T. J. McDougall, *J. Fluid Mech.* **126** (1983), 379.
20. P. G. Baines and A. E. Gill, *J. Fluid Mech.* **37** (1969), 289.

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HODOGRAPH TRANSFORMATION IN CONSTANTLY INCLINED TWO-PHASE MFD FLOWS

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Hodograph transformation is employed for steady, plane, viscous, incompressible constantly inclined two-phase MFD flows and a partial differential equation of second order obtained which is used to find the solution for vortex flow.

1. INTRODUCTION

Multiphase fluid phenomena are of extreme importance in various fields of science and technology such as geophysics, nuclear engineering, chemical engineering etc. In recent years, considerable attention has been paid to the study of the multiphase fluid flow system in non-rotating or rotating frames of reference. Multiphase fluid systems are concerned with the motion of a liquid or gas containing immiscible inert particles. Of all multiphase fluid systems observed in nature, blood flow, flow in rocket chamber, dust in gas cooling systems to enhance heat transfer process, movement of inert particles in atmosphere and sand or other suspended particles in sea beaches are the most common examples Naturally, studies of these systems are mathematically interesting and physically useful. The presence of particles in a homogeneous fluid makes the dynamical study of flow problems quite complicated. However, these problems are usually investigated under various simplifying assumptions.

Saffman[1] has formulated the equations of motion of a dusty fluid which is represented in terms of large number density $N(x, t)$ of very small spherical inert particles whose volume concentration is small enough to be neglected. It is assumed that the density of the dust particles is large when compared with the fluid density so that the mass concentration of the particles is an appreciable fraction of unity. In this formulation, Saffman also assumed that the individual particles of dust are so small that stokes' law of resistance between the particles and the fluid remains valid. Using the model of Saffman, several authors including Michael and Miller[2], Liu[3], Debnath and Basu[4] and S N. Singh *et al.*[5], have investigated various aspects of hydrodynamics and hydromagnetic two-phase fluid flows in non-rotating system.

Transformation techniques have become some of the powerful methods for solving non-linear partial differential equations. Amongst many, the hodograph transfor-

mation has gained considerable success on fluid dynamics problems. Ames[6] has given an excellent survey of this method together with its application in various other fields. Chandna *et al.*[8] have used the hodograph transformation for steady MFD flows. Also Singh *et al.*[10] have used hodograph transformation in steady rotating MHD flows and obtained some solutions.

In this paper, hodograph transformation is employed for steady, plane, viscous incompressible constantly inclined two-phase MFD flows and a partial differential equation of second order obtained which is used to find the solution for vortex flow.

2. Basic Equations

The basic equations of motion governing the steady flow of a dusty, incompressible, viscous fluid with infinite electrical conductivity in the presence of magnetic field are

$$\operatorname{div} \bar{u} = 0 \quad \text{...(2.1)}$$

$$\rho\,[(\bar{u}\ \text{grad})\ \bar{u}] = -\ \text{grad}\ p + \mu\ \text{curl}\ \overline{H} \times \overline{H} + KN\,(\bar{v} - \bar{u}) + \eta\, \nabla^2 \bar{u} \quad \text{...(2.2)}$$

$$\text{curl}\,(\bar{u} \times \overline{H}) = \bar{0} \quad \text{...(2.3)}$$

$$\operatorname{div}\,(N\,\bar{v}) = 0 \quad \text{...(2.4)}$$

$$m\,(\bar{v}.\ \text{grad})\ \bar{v} = K\,(\bar{u} - \bar{v}) \quad \text{...(2.5)}$$

$$\operatorname{div} \overline{H} = 0 \quad \text{...(2.6)}$$

where $\bar{u}$, $\bar{v}$, $\overline{H}$, p, ρ, η, μ are fluid velocity vector, dust velocity vector, magnetic field vector, fluid pressure, fluid density, kinematic coefficient of viscosity and magnetic permeability respectively; m is the mass of each dust particle, N the number density of dust particles and K the Stokes' resistance coefficient for the particles.

The situation for which the velocity of fluid and dust particles are everywhere parallel, is defined as[11]

$$\bar{v} = \frac{\alpha}{N}\ \bar{u} \quad \text{...(2.7)}$$

where α is some scalar satisfying

$$\bar{u}.\ \text{grad}\ \alpha = 0 \quad \text{...(2.8)}$$

which implies that α is a constant on the fluid streamlines.

Introducing vorticity function, current density function and Bernoulli function defined, respectively, by

$$\xi = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \qquad \ldots(2.9)$$

$$\Omega = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \qquad \ldots(2.10)$$

$$B = p + \tfrac{1}{2} \rho\, U^2 \qquad \ldots(2.11)$$

where $U^2 = u_1^2 + u_2^2$, the system of eqns. (2.1) – (2.6) can be replaced by the following system

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \qquad \ldots(2.12)$$

$$\eta \frac{\partial \xi}{\partial y} - \rho \xi u_2 + \mu \Omega H_2 - K(\alpha - N) u_1 = - \frac{\partial B}{\partial x} \qquad \ldots(2.13)$$

$$\eta \frac{\partial \xi}{\partial x} - \rho \xi u_1 + \mu \Omega H_1 + K(\alpha - N) u_2 = \frac{\partial B}{\partial y} \qquad \ldots(2.14)$$

$$u_1 H_2 - u_2 H_1 = f \text{ (arbitrary constant)} \qquad \ldots(2.15)$$

$$\frac{m\alpha}{N}\left[\frac{\alpha}{N}\left(u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_2}{\partial y}\right) + u_1\left(u_1 \frac{\partial}{\partial x}\left(\frac{\alpha}{N}\right) + u_2 \frac{\partial}{\partial y}\left(\frac{\alpha}{N}\right)\right)\right]$$
$$= K\left(\frac{\alpha}{N} - 1\right) u_1 \qquad \ldots(2.16)$$

$$\frac{m\alpha}{N}\left[\frac{\alpha}{N}\left(u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y}\right) + u_2\left(u_1 \frac{\partial}{\partial x}\left(\frac{\alpha}{N}\right) + u_2 \frac{\partial}{\partial y}\left(\frac{\alpha}{N}\right)\right)\right]$$
$$= K\left(\frac{\alpha}{N} - 1\right) u_2 \qquad \ldots(2.17)$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0. \qquad \ldots(2.18)$$

The advantage of this system over the original system is that the order of the partial differential equation is decreased.

We now consider constantly inclined plane flows and let α_0 denote the constant non-zero angle between $\bar{u}$ and $\bar{H}$. The vector and scalar products of $\bar{u}$ and $\bar{H}$, using the diffusion equation (2.15), yield

$$\begin{aligned} u_1 H_2 - u_2 H_1 &= UH \sin \alpha_0 = f \\ u_1 H_1 + u_2 H_2 &= UH \cos \alpha_0 = f \cot \alpha_0 \end{aligned} \qquad \ldots(2.19)$$

where

$$H = \sqrt{H_1^2 + H_2^2}\,.$$

Solving (2.19), we get

$$H_1 = \frac{f}{U^2}(C\,u_1 - u_2),\; H_2 = \frac{f}{U^2}(C\,u_2 + u_1) \qquad \text{...(2.20)}$$

where $C = \cot\alpha_0$ is a known constant for a prescribed constantly inclined non-aligned flow.

Using (2.20) in the system of equations (2.9) – (2.18), we have

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \qquad \text{...(2.21)}$$

$$\eta\,\frac{\partial \xi}{\partial x} - \rho\xi\,u_1 + \frac{\mu\Omega f}{U^2}(C\,u_2 + u_1) - K(\alpha - N)\,u_1$$

$$= -\frac{\partial B}{\partial x} \qquad \text{...(2.22)}$$

$$\eta\,\frac{\partial \xi}{\partial x_1} - \rho\xi\,u_2 + \frac{\mu\Omega f}{U^2}(C\,u_1 - u_2) + K(\alpha - N)\,u_2 = \frac{\partial B}{\partial y} \qquad \text{...(2.23)}$$

$$\frac{m\alpha}{N}\left[\frac{\alpha}{N}\left(u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y}\right) + u_1\left(u_1\frac{\partial}{\partial x}\left(\frac{x}{N}\right)\right.\right.$$

$$\left.\left. + u_2\frac{\partial}{\partial y}\left(\frac{\alpha}{N}\right)\right)\right] = K\left(\frac{\alpha}{N} - 1\right)u_1 \qquad \text{...(2.24)}$$

$$\frac{m\alpha}{N}\left[\frac{\alpha}{N}\left(u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y}\right) + u_2\left(u_1\frac{\partial}{\partial x}\left(\frac{\alpha}{N}\right) + u_2\frac{\partial}{\partial y}\left(\frac{\alpha}{N}\right)\right)\right]$$

$$= K\left(\frac{\alpha}{N} - 1\right)u_2 \qquad \text{...(2.25)}$$

$$(u_2^2 - u_1^2 + 2u_1u_2)\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) + (Cu_2^2 - Cu_1^2 + 2u_1u_2)$$

$$\left(\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial y}\right) = 0 \qquad \text{...(2.26)}$$

$$\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} = \xi \qquad \text{...(2.27)}$$

$$\frac{\partial}{\partial x}\left(\frac{C\,u_2 + u_1}{U^2}\right) - \frac{\partial}{\partial y}\left(\frac{C\,u_1 - u_2}{U^2}\right) = \frac{\Omega}{f}. \qquad \text{...(2.28)}$$

Let the flow variables $u_1(x, y)$, $u_2(x, y)$ be such that, in the flow region under consideration, the Jacobian

$$J = \frac{\partial(u_1, u_2)}{\partial(x, y)} \text{ satisfies } 0 < |J| < \infty.$$

In such a case, we consider x and y as functions of u_1 and u_2 such that the following relations hold true :

$$\frac{\partial u_1}{\partial x} = J \frac{\partial y}{\partial u_2}, \quad \frac{\partial u_1}{\partial y} = - J \frac{\partial x}{\partial u_1}$$

$$\frac{\partial u_2}{\partial x} = - J \frac{\partial y}{\partial u_1}, \quad \frac{\partial u_2}{\partial y} = J \frac{\partial x}{\partial u_1}. \qquad \text{...(2.29)}$$

Employing transformation equation (2.29) in (2 21) and (2.26), we get

$$\frac{\partial x}{\partial u_1} + \frac{\partial y}{\partial u_2} = 0 \qquad \text{...(2 30)}$$

$$(C u_1^2 - C u_2^2 - 2u_1 u_2) \left(\frac{\partial x}{\partial u_1} - \frac{\partial y}{\partial u_2} \right)$$

$$+ (u_1^2 - u_2^2 + 2C u_1 u_2) \left(\frac{\partial x}{\partial u_2} + \frac{\partial y}{\partial u_1} \right) = 0. \qquad \text{...(2.31)}$$

The equation of continuity implies the existence of stream function $\psi(x, y)$ so that

$$\frac{\partial \psi}{\partial x} = - u_2, \frac{\partial \psi}{\partial y} = u_1. \qquad \text{...(2.32)}$$

Likewise, equation (2.30) implies the existence of a function $L(u_1, u_2)$ called the Legendre transform of the stream function $\psi(x, y)$ such that

$$\frac{\partial L}{\partial u_1} = - y, \frac{\partial L}{\partial u_2} = x. \qquad \text{...(2.33)}$$

Employing (2.33) in (2.31), we have

$$(u_2^2 - u_1^2 - 2C u_1 u_2) \frac{\partial^2 L}{\partial u_1^2} + (2C u_1^2 - 2C u_2^2 - 4u_1 u_2)$$

$$\times \frac{\partial^2 L}{\partial u_1 \partial u_2} + (2C u_1 u_2 + u_1^2 - u_2^2) \frac{\partial^2 L}{\partial u_2^2} = 0.$$

$$\text{...(2.34)}$$

Now introducing the polar coordinate (U, θ) in the hodograph plane i.e. the (u_1, u_2) plane through the relation :

$$u_1 = U \cos \theta, \quad u_2 = U \sin \theta$$

equation (2.34) gets transformed into

$$\frac{\partial^2 L}{\partial U^2} - \frac{2C}{U} \frac{\partial^2 L}{\partial U \partial \theta} - \frac{1}{U^2} \frac{\partial^2 L}{\partial \theta^2} - \frac{1}{U} \frac{\partial L}{\partial U} + \frac{2C}{U^2} \frac{\partial L}{\partial \theta} = 0 \qquad \text{...(2.35)}$$

where θ is the inclination of vector field $\bar{u}$.

3. Vortex Flow

A solution of (2.35) is given by

$$L = B_2 U^2 + (A_1 \cos\theta + B_1 \sin\theta)\, U$$

$$= B_2 (u_1^2 + u_2^2) + A_1 u_1 + B_1 u_2 \qquad \text{...(3.1)}$$

where A_1, B_1 and B_2 are arbitrary constant and $B_2 \neq 0$. In this case,

$$x = \frac{\partial L}{\partial u_2} = 2B_2 u_2 + B_1,\ y = -\frac{\partial L}{\partial u_1} = -(2B_2 u_1 + A_1) \qquad \text{...(3.2)}$$

and therefore the velocity field is given by

$$u_1 = -\frac{y + A_1}{2B_2},\ u_2 = \frac{x - B_1}{2B_2}. \qquad \text{...(3.3)}$$

These relations represent a circulatory flow.

From (2.20), we get

$$H_1 = \frac{-2B_2 f[(x - B_1) + C(y + A_1)]}{(X - B_1)^2 + (y + A_1)^2}$$

and

$$H_2 = \frac{2B_2 f[C(x - B_1) - (y + A_1)]}{(X - B_1)^2 + (y + A_1)^2}. \qquad \text{...(3.4)}$$

The vorticity ξ and current density Ω can be expressed as

$$\xi = \frac{1}{B_2},\ \Omega = 0. \qquad \text{...(3.5)}$$

From the integrability condition for B with the use of (2.13) and (2.14) and [(3.3)–(3.5)], we obtain

$$(x - B_1)\frac{\partial}{\partial x}(N - \alpha) + (y + A_1)\frac{\partial}{\partial y}(N - \alpha) + 2(N - \alpha) = 0. \qquad \text{...(3.6)}$$

Solving (3.6), the number density of dust particles $N(x, y)$ is given by

$$N = \frac{C_1}{(x - B_1)(y + A_1)} + \alpha \qquad \text{...(3.7)}$$

where C_1 is an arbitrary constant. From equation (2.8) and (3.3), we obtain

$$\alpha = C_2[(x - B_1)^2 + (y + A_1)^2] \qquad \text{...(3.8)}$$

where C_2 is an arbitrary constant.

Hence

$$N = \frac{C_1}{(x - B_1) + (y + A_1)} + C_2[(x - B_1)^2 + (y + A_1)^2]. \qquad \text{...(3.9)}$$

Using (3.3) – (3.5) and (3.7) in (2.22) and (2.23) and solving, we get

$$B = \frac{\rho}{4B_2^2}\,[(x - B_1)^2 + (y + A_1)^2] + \frac{K\,C_1}{2B_2}\ln\frac{x - B_1}{y + A_1} + C_3 \qquad \ldots(3.10)$$

where C_3 is an arbitrary constant. The pressure P is given by

$$P = C_3 + \frac{\rho}{8B_2^2}\,[(x - B_1)^2 + (y + A_1)^2] + \frac{K\,C_1}{2B_2}\ln\frac{x - B_1}{y + A_1}. \qquad \ldots(3.11)$$

In this case the streamlines are given by

$$(x - B_1)^2 + (y + A_1)^2 = \text{constant}$$

which are concentric circles. Summing up, we have :

Theorem 1—If the dust particle is everywhere parallel to the fluid velocity in the steady, plane, constantly inclined MFD flow of an incompressible, viscous, two phase fluid, then the streamlines are concentric circles and the dust particle number density is given by (3.9). Also the velocity, the magnetic field, the vorticity, the current density and the pressure are given by (3.3), (3.4), (3.5) and (3.11) respectively.

4. Conclusion

There are very few exact solutions of two-phase MFD flows. The mathematical complexity of the equations governing the flow of an electrically conducting has prohibited a thorough analysis. To reduce some of the complexity, it becomes necessary to make certain assumptions about the inherent properties of the two-phase fluid. Furthermore, all the methods of analysis to this date require that we impose some restrictions on the angle between velocity field vector and magnetic field vector. In the present work, Saffman model for infinitely conducting two-phase fluid flow considering constant angle between u and H, called constantly inclined flow, is taken and exact solutions of physical importance are obtained applying hodograph transformation. Although the scope of the present work is limited, it is believed that by using the approach of this paper and the exact solution obtained, work towards boundary value problem of practical importance can be pursued.

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References

1. P. G. Saffman, *J. Fluid Mech.* **13** (1962), 120.
2. D. H. Michael and D. A. Miller, *Mathematika* **13** (1966), 97.
3. J. T. C. Liu, *Astronoutika Acta* **13** (1967), 369.

4. L. Debnath and U. Basu, *Nuovo Cimento* 28B (1975), 349.
5. S. N. Singh and R. Babu, *Ap. Sp. Sci.* **104**, (1984), 285.
6. W. F. Ames, *Non-linear Partial Differential Equation in Engineering*. Academic Press, New York, 1965.
7. R. M. Barron and O. P. Chandna, *J. Engng. Math*. **15** (3) (1981), 210.
8. O. P. Chandna and M. R. Garg, *Int J. Engng. Sci.* **17** (1979), 251.
9. O. P. Chandna and H. Toews, *Quart. Appl. Math.* **35** (1977), 331.
10. S. N. Singh, H. P. Singh and R. Babu, *Ap. Sp. Sci.* **106** (1984), 231.
11. R. M. Barron, *Tensor, N. S.* **31**, (1977), 271.

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THERMAL STABILITY OF A FLUID LAYER IN A VARIABLE GRAVITATIONAL FIELD

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The instability of a heated layer of a viscous fluid confined between two horizontal planes and subjected to a variable gravitational field varying spatially with height is investigated. For a layer confined between two stress-free boundaries, irrespective of whether gravitational acceleration is increasing or decreasing with height, it is shown that (i) the principle of exchange of stabilities is valid when the layer is heated from below, and (ii) the layer is stable when it is heated from above. In the latter case, the complex growth rate of an arbitrary oscillatory mode lies outside of a circle whose radius depends on the wavelength of the mode and the Prandtl number of the fluid but not on the Rayleigh number. The underlying characteristic value problem is solved approximately for a linearly varying gravity field and stress-free boundaries. It is found that in the case of a layer heated from below, gravity increasing upward is a destabilizing influence.

1. Introduction

The importance of convection currents in our environment can scarcely be overestimated; we need only look to our immediate environment and note that the circulation of the Earth's atmosphere could not be explained without reference to convective motions induced by solar heating. However, the idealization of uniform gravity assumed in the theoretical investigations, although valid for laboratory purposes, can scarcely be justified for large-scale convection phenomena occurring in the atmosphere, the ocean or the mantle of the earth. It then becomes imperative to consider gravity as a variable quantity varying with distance from a reference point or surface.

Although Gresho and Sani[3] have considered a time-varying gravitational field in the convection problem of a horizontal fluid layer, the problem of a spatially varying gravitational field in the plane layer case still remains open. Pradhan and Patra[4] considered the problem of onset of thermal instability in a cylindrical shell of self-gravitating fluid heated internally and from below i.e. from the inside boundary. In case of a shell heated from below, the narrow-gap approximation led to the plane layer problem with the gravity field varying linearly with height from the bottom surface. This prompted Pradhan and Samal[5] to consider the plane layer problem

under a spatially-varying gravity field, neglecting viscosity. They found, among other things, that if gravity remained downward (upward) throughout the flow domain, neutral modes did not exist. A sufficient condition for stability of a layer heated from above was that gravity remain directed downward over a sufficiently large part of the flow domain. However, a sufficient condition for instability of a layer heated from below was that, besides remaining directed downward, the gravity profile must have concave curvature throughout the flow domain. There also existed a circle limiting the growth rates of an arbitrary oscillatory mode of the system. These results, which are indicative of the behaviour expected in the limit of small viscosity in the fluid, are here extended to the case of finite viscosity.

2. Perturbation Equations

Consider a layer of an incompressible, viscous fluid statically confined between two horizontal boundaries $z = 0$ and $z = d$ which are maintained at constant temperatures T_0 and T_1 respectively. The governing equations for flow and temperature, under the Boussinesq approximation, are

$$\rho_0 \left(\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u} \right) = - \nabla p - \rho_0 \, g(z) \, \bar{k} \, (1 - \alpha (T - T_0)) + \mu \nabla^2 \bar{u} \qquad \ldots(1)$$

$$\nabla \cdot \bar{u} = 0 \qquad \ldots(2)$$

and

$$\frac{\partial T}{\partial t} + \bar{u} \cdot \nabla T = \kappa \nabla^2 T \qquad \ldots(3)$$

where ρ is the density of the fluid, $\bar{u} = (u, v, w)$ the fluid velocity, $g(z)$ the gravitational acceleration in a coordinate system $Oxyz$ having Oz vertical and Ox, Oy in a horizontal plane, k is a unit vetor in the vertically upward direction; p the pressure in the fluid; μ the dynamic viscocity; T the obsolute temperature; κ the thermal diffusivity and ρ_0 the density at temperature T_0.

In the equilibrium state heat transport is by conduction alone and there are no velocities in the fluid. The solution of eqns. (1)–(3) is easily obtained in this case as the solution of (the basic state is denoted by an over bar).

$$\bar{u} = 0 \qquad \ldots(4)$$

$$\nabla \bar{p} = g \, \rho_0 \, \bar{k} \, \{1 - \alpha (\bar{T} - T_0)\} \qquad \ldots(5)$$

$$\nabla^2 \bar{T} = 0. \qquad \ldots(6)$$

The basic temperature $\bar{T}$ is assumed to depend only on z, so we can integrate (6) to obtain

$$\bar{T} = T_0 + \beta z \qquad \ldots(7)$$

where $\beta = (T_0 - T_1)/d$ is negative, since this is the solution which gives $T = T_0$, T_1 at $z = 0$, d.

Writing

$$p = \bar{p} + p', \; T = \bar{T} + T', \; \rho = \bar{\rho} + \rho', \; \bar{u} = \bar{u}' \qquad \ldots(8)$$

where p', T', ρ' and $\bar{u}'$ are small perturbations with respect to which the equations are linearized and using the Boussinesq approximation, we obtain

$$\frac{\partial \bar{u}'}{\partial t} = \frac{1}{\rho_0} \nabla p' + \nu \nabla^2 \bar{u} + g \alpha T' \bar{k} \qquad \ldots(9)$$

$$\nabla \cdot \bar{u}' = 0 \qquad \ldots(10)$$

and

$$\frac{\partial T'}{\partial t} + \beta w = \kappa \nabla^2 T' \qquad \ldots(11)$$

where ν is the kinematic viscosity.

Seeking solutions in terms of normal modes whose dependence on x, y and t is given by

$$\exp i (k_n x + k_y y + \sigma t) \qquad \ldots(12)$$

and eliminating p', u' and v' from the resulting equations, we have

$$\nu (D^2 - k^2)\left(D^2 - k^2 - \frac{i\sigma}{\nu} \right) w = k^2 g \alpha T' \qquad \ldots(13)$$

and

$$i \sigma T' - \kappa (D^2 - k^2) T' = - \beta w \qquad \ldots(14)$$

where D denotes differentiation with respect to z and $k^2 = k_X^2 + k_Y^2$.

Assuming the boundaries to be perfectly heat conducting and either rigid or free we have

$$T' = 0 = w \text{ for } z = 0 \text{ and } d$$

and either $Dw = 0$ (in case of a rigid boundary)

or $D^2 w = 0$ (in case of a free boundary) $\qquad \ldots(15)$

Introducing the non-dimensional variables

$$z_* = z/d, \quad a = kd, \quad \sigma_* = \frac{i \sigma d^2}{\nu},$$

$$w_* = w\, d/k, \; D_* = dD, \; \theta_* = T/\beta d \qquad \ldots(16)$$

and $g(z) = g_0 \gamma(z_*)$ where g_0 is the value of $g(z)$ at $z = 0$, we can write the eqns. (13) and (14), on suppressing the star subscripts, as

$$(D^2 - a^2 - \sigma)(D^2 - a^2) w = - Ra^2 \gamma(z) \theta \quad \ldots(17)$$

and

$$(D^2 - a^2 - P_r) \theta = w \quad \ldots(18)$$

where

$$R = \frac{-g_0 \alpha \beta d^4}{\nu \kappa} \quad \ldots(19)$$

and

$$P_r = \nu/\kappa \quad \ldots(20)$$

are the non-dimensional Rayleigh and Prandtl numbers, respectively.

The boundary conditions reduce to $w = 0 = \theta$ $(z = 0, 1)$ and either

$$Dw = 0 \quad \text{(at a rigid boundary)}$$

or

$$D^2 w = 0 \quad \text{(at a free boundary).} \quad \ldots(21)$$

Thus, for a given a, R and P_r, the equations (17) and (18) together with the boundary conditions (21) constitute an eigenvalue problem for σ and the system is unstable, neutral or stable according as the real part of σ, namely, σ_r is positive, zero or negative, respectively.

3. The Principle of Exchange of Stabilities

For fixed, stress-free boundaries at $z = 0, 1$ we have the kinematic boundary condition as

$$w = D^2 w = 0 \ (z = 0, 1). \quad \ldots(22)$$

Since the thermal boundary condition is

$$\theta = 0 \ (z = 0, 1) \quad \ldots(23)$$

the boundary condition (22) can be expressed in terms of θ by means of (18) and (23) as

$$D^2 \theta = D^4 \theta = 0 \ (z = 0, 1). \quad \ldots(24)$$

So finally

$$\theta = D^2 \theta = D^4 \theta = 0 \ (z = 0, 1). \quad \ldots(25)$$

Combining eqns. (17) and (18), we have

$$(D^2 - a^2 - P_r \sigma)(D^2 - a^2 - \sigma)(D^2 - a^2) \theta = - Ra^2 \gamma(z) \theta \quad \ldots(26)$$

which must be considered together with the boundary conditions (25).

Multiplying eqn. (26) by θ^*, the complex conjugate of θ, and integrating the resulting equation over the range of z and making use of the boundary conditions (25), we have

$$P_r \sigma^2 \int_0^1 (| D\theta |^2 + a^2 | \theta |^2) dz + \sigma (1 + P_r) \int_0^1 | (D^2 - a^2) \theta |^2 dz$$

$$+ \int_0^1 (| D^3 \theta |^2 + 3a^2 | D^2 \theta |^2 + 3a^4 | D\theta |^2$$

$$+ a^6 | \theta |^2) dz - Ra^2 \int_0^1 \gamma (z) | \theta |^2 dz = 0. \qquad \text{...(27)}$$

Now, for a neutral mode, we must have $\sigma = i\sigma_i$ with σ_i real, and then the real and imaginary parts of (27) give

$$P_r \sigma_i^2 \int_0^1 (| D\theta |^2 + a^2 | \theta |^2) dz - \int_0^1 (| D^3 \theta |^2 + 3a^2 | D^2 \theta |^2$$

$$+ 3a^4 | D\theta |^2 + a^6 | \theta |)^2 dz + Ra^2 \int_0^1 \gamma (z) | \theta |^2 dz = 0 \qquad \text{...(28)}$$

and

$$(1 + P_r) \sigma_i \int_0^1 | (D^2 - a^2) \theta |^2 dz = 0. \qquad \text{...(29)}$$

Equation (29) shows that $\sigma_i = 0$ and then (28) shows that the principle of exchange of stabilities is valid provided the neutral state exists, that is,

$$\int_0^1 (| D^3 \theta |^2 + 3a^2 | D^2 \theta^2 |^2 + 3a^4 | D\theta |^2 + a^6 | \theta |^2) dz$$

$$= Ra^2 \int_0^1 \gamma (z) | \theta |^2 dz. \qquad \text{...(30)}$$

Equation (30) shows that, since $\gamma (z) > 0$, a necessary condition for the existence of neutral states is that

$$R > 0. \qquad \text{...(31)}$$

If R is negative, there is no neutral state and a random disturbance will either be damped or amplified.

4. A Sufficient Condition for Stability

Since $\sigma \neq 0$ for R negative in the case of stress-free boundaries, multiplying eqn. (27) throughout by σ^*, the complex conjugate of σ and dividing by $| \sigma |^2$ we have

$$P_r\,\sigma \int_0^1 (|D\theta|^2 + a^2|\theta|^2)\,dz + (1+P_r)\int_0^1 |(D^2-a^2)\theta|^2\,dz$$

$$+\frac{\sigma^*}{|\sigma|^2}\int_0^1 (|D^3\theta|^2 + 3a^2|D^2\theta|^2 + 3a^4|D\theta|^2 + a^6|\theta|^2)\,dz$$

$$-\frac{Ra^2\,\sigma^*}{|\sigma|^2}\int_0^1 \gamma(z)\,|\theta|^2\,dz = 0. \qquad ...(32)$$

Separating the real and imaginary parts of eqn. (32), we have

$$P_r\,\sigma_r \int_0^1 (|D\theta|^2 + a^2|\theta|^2)\,dz + (1+P_r)\int_0^1 |(D^2-a^2)\theta|^2\,dz$$

$$+\frac{\sigma_r}{|\sigma|^2}\int_0^1 (|D^3\theta|^2 + 3a^2|D^2\theta|^2 + 3a^4|D\theta|^2 + a^6|\theta|^2)\,dz$$

$$-Ra^2\,\frac{\sigma_r}{|\sigma|^2}\int_0^1 \gamma(z)\,|\theta|^2\,dz = 0 \qquad ...(33)$$

and

$$P_r\,\sigma_i \int_0^1 (|D\theta|^2 + a^2|\theta|^2)\,dz - \frac{\sigma_i}{|\sigma|^2}\int_0^1 (|D^3\theta|^2$$

$$+ 3a^2|D^2\theta|^2 + 3a^4|D\theta|^2 + a^6|\theta|^2)\,dz$$

$$+ Ra^2\,\frac{\sigma_i}{|\sigma|^2}\int_0^1 \gamma(z)\,|\theta|^2\,dz = 0. \qquad ...(34)$$

Equation (34) shows that if $R < 0$, then $\sigma_r < 0$ provided $\gamma(z) > 0$ over most of the layer thus implying stability for the configuration.

5. A Circular Exclusion Theorem for Oscillatory Modes

Consider an arbitrary oscillatory mode of the system in the case of stress-free boundaries. For such modes, we have $\sigma_i \neq 0$. Hence eqn. (34) can be put as

$$\int_0^1 (|D^3\theta|^2 + 3a^2|D^2\theta|^2 + 3a^4|D\theta|^2 + a^6|\theta|^2$$

$$- Ra^2\gamma(z)\,|\theta|^2)\,dz - P_r\,\sigma^2\int_0^1 (|D\theta|^2 + a^2|\theta|^2)\,dz = 0. \qquad ...(35)$$

This can be rewritten as

$$\int_0^1 (\,|\,D^3\,\theta\,|^2 + 3a^2\,|\,D^2\,\theta\,|^2 + 2a^4\,|\,D\theta\,|^2)\,dz$$

$$+ (a^4 - P_r\,|\,\sigma\,|^2)\int_0^1 (\,|\,D\theta\,|^2 + a^2\,|\,\theta\,|^2)\,dz$$

$$= Ra^2 \int_0^1 \gamma\,(z)\,|\,\theta\,|^2\,dz. \qquad \ldots(36)$$

Hence if $R < 0$, then we must have

$$\sigma_r^2 + \sigma_i^2 > a^4/P_r. \qquad \ldots(37)$$

Hence the complex growth rate of an arbitrary oscillatory mode (in the case of a layer heated from above in a variable gravitational field directed downward) must lie outside of a circle whose radius depends on the wavelength of the mode as well as the Prandtl number of the fluid (but not on the Rayleigh number of the configuration).

6. The Problem of a Layer with Free Boundaries

We solve the characteristic value problem consisting of (17), (18) and (21) for free upper and lower boundaries for the case of a linearly varying gravity field

$$\gamma\,(z) = 1 + Mz > 0. \qquad \ldots(38)$$

Since the principle of exchange of stabilities is true in this case and the marginal state is stationary, the equations to be solved are

$$(D^2 - a^2)^2\,w = -\,Ra^2\,(1 + Mz)\,\theta \qquad \ldots(39)$$

and

$$(D^2 - a^2)\,\theta = w \qquad \ldots(40)$$

together with the boundary conditions

$$w = D^2\,w = \theta = 0,\ (z = 0,\ 1) \qquad \ldots(41)$$

Eliminating w from eqns. (39) and (40) we get

$$(D^2 - a^2)^3\,\theta = -\,Ra^2\,(1 + Mz)\,\theta. \qquad \ldots(42)$$

The kinematic boundary condition can be expressed in terms of θ by eqn. (24). In view of (39) and (40) this gives

$$D^6\,\theta = D^8\,\theta = D^4\,\theta = \ldots = D^{2n}\,\theta = 0,\ (z = 0,\ 1). \qquad \ldots(43)$$

So θ must satisfy

$$\theta = D^2\,\theta = D^4\,\theta = \ldots = D^{2n}\,\theta = 0,\ (z = 0,\ 1). \qquad \ldots(44)$$

We have to solve eqn. (42) with the boundary condition (44). Rewriting eqn. (42) in the manner

$$(D^2 - a^2)^3 \theta = (1 + Mz) \psi \qquad ...(45)$$

and

$$\psi = - Ra^2 \theta \qquad ...(46)$$

we expand ψ and θ in the forms

$$\psi = \sum_{m=1}^{\infty} A_m \sin m \pi z \text{ and } \theta = \sum_{m=1}^{\infty} \frac{-A_m \sin m\pi z}{Ra^2} \qquad ...(47)$$

where θ is the solution of the equation

$$(D^2 - a^2)^3 \theta = (1 + Mz) \sum_{m=1}^{\infty} A_m \sin m \pi z \qquad ...(48)$$

which satisfies the boundary conditions (44).

Inserting the expansions (47) in eqn. (46) and multiplying the resulting equation by $\sin n\pi z$ and integrating from 0 to 1 gives a set of homogeneous linear algebraic equations in the A_m. The requirement that the constants A_m in the resulting algebraic equations are not all zero leads to the secular equation. An approximation to the solution can be obtained by setting the determinant formed by the elements in the first n-rows and n-columns of the secular determinant equal to zero, successive approximations being obtained as n increases. The calculation can be done for various values of M, in each case the value of R which corresponds to a particular value of a being found by solving an algebraic equation. The minimum value of R with respect to a can then be found out for each M.

A first approximation to the solution of the secular equation is obtained by setting the (1, 1)-element of the matrix equal to zero and ignoring all the others. This corresponds to the choice of $\sin \pi z$ as a trial function for θ. As Chandrasekhar[1,2] has shown, very good approximation are obtained for $n = 1 = m$ and the solution converges rapidly for increasing values of m and n. Setting the (1, 1)-element of the secular determinant equal to zero, we find that

$$\tfrac{1}{2} (\pi^2 + a^2) \frac{1}{Ra^2} = \tfrac{1}{4} M + \tfrac{1}{2}. \qquad ...(49)$$

This gives

$$R = \frac{2}{2 + M} \frac{(\pi^2 + a^2)^3}{a^2}. \qquad ..(50)$$

We can get a better approximate solution by taking the first two terms in the expansion for θ and w. In this case, the secular determinant reduces to the form

$$\begin{vmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{vmatrix} = 0 \qquad ...(51)$$

where

$$S_{11} = -\frac{(\pi^2 + a^2)^3}{2Ra^2} + \tfrac{1}{4} M + \tfrac{1}{2}$$

$$S_{12} = -\frac{8M}{\pi^2 + a^2}\left\{\frac{a^2 + 13\pi^2}{9\pi^2} + \frac{3\pi^2 (2a^4 + 8a^2\pi^2 + 9\pi^4)}{(a^2 + 4\pi^2)(a^2 + \pi^2)^2}\right\}$$

$$S_{21} = \frac{8M(8\pi^2 - a^2)}{9\pi^2(\pi^2 + a^2)} + \frac{M\pi^2(\pi^2 + a^2)}{a^2(4\pi^2 + a^2)^2}\left[\frac{(4\pi^2 - 3a^2)}{4\pi^2 + a^2}\right.$$

$$\left.\{1 - \cosh a + \sinh a\,(1 + \cosh a)\} + \frac{4a(1 + \cosh a)}{\sinh a}\right]$$

$$S_{22} = -\frac{(4\pi^2 + a^2)^3}{2Ra^2} + \tfrac{1}{4} M + \tfrac{1}{2}. \qquad \text{...(52)}$$

Equation (51) is quadratic in R. The minimum positive value of R with respect to a for each M can be calculated from this equation.

7. Discussion

Numerical results summarized in Figs. 1—3 were obtained by evaluating (51). For comparison purposes, separate curves (dotted ones) have been drawn using (50). It is seen that a variable gravitational field with gravity increasing linearly upward has a destabiling effect (Fig. 1), the neutral stability curves shifting progressively downward

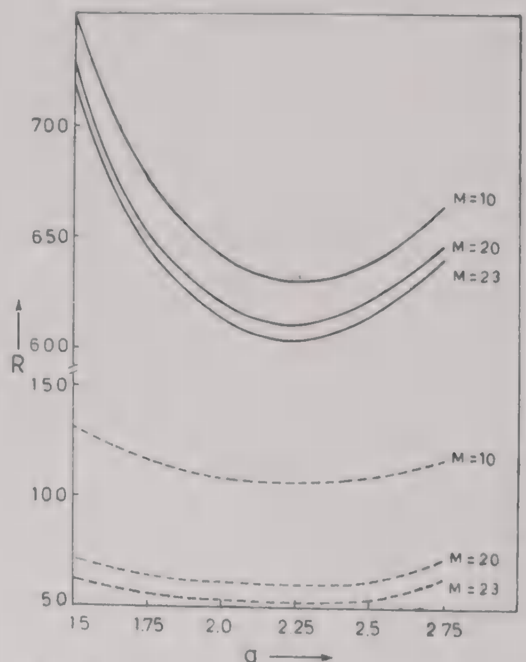

Fig. 1. Neural stability curves for different values of the gravity parameter.

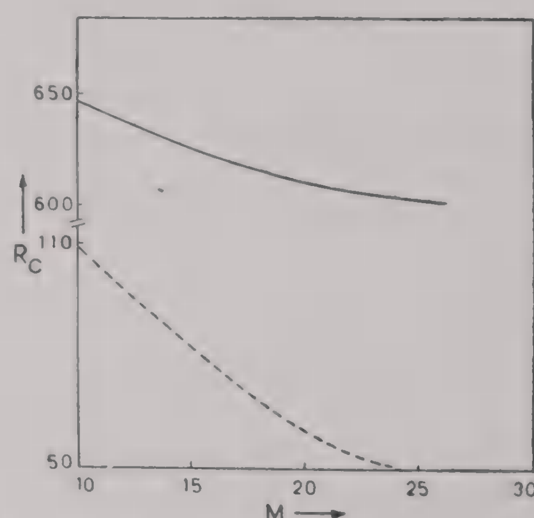

Fig. 2 Behaviour of the critical Rayleigh number with increase in gravity parameter.

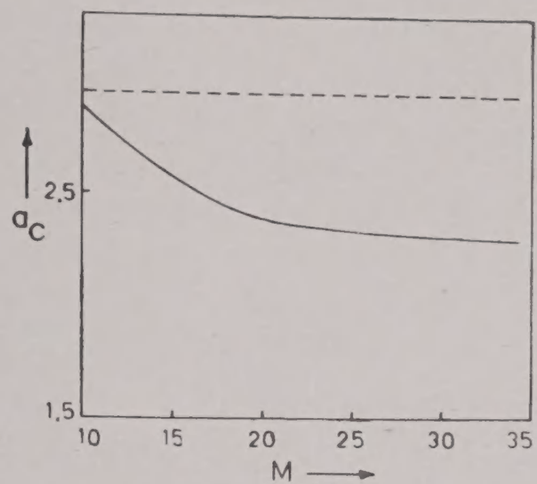

FIG. 3 Behaviour of critical wave number a_c with increase in gravity parameters.

as the gravity parameter increases. The monotonic increase in the destabilizing effect with increase in the gravity parameter is further brought out in Fig. 2. Figure 3 shows that the critical wave-number decreases monotonically with increase in the gravity parameter.

We expect similar behaviour for other boundary conditions.

REFERENCES

1. S. Chandrasekhar, *Mathematika* **1** (1954), 5–13.
2. S. Chandrasekhar, *Hydrodynamic and Hydramagnetic Stability*. Clarendon Press, Oxford, 1961.
3. P. M. Gresho and R. L. Sani, *J. Fluid Mech.* **40** (1970), 783–806.
4. G. K. Pradhan and B. Patra, *Proc. Indian Acad. Sci. A: Math. Sci.* (in press).
5. G. K. Pradhan and P. C. Samal, *J. Math. Anal. Applic.* **122** (1987), 487–95.

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